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Hyers-Ulam-Rassias Stability of Quartic Functional Equation in Paranormed Spaces

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Abstract: *Throughout this paper , we investigate the Hyers-Ulam stability of Quartic functional equation $f(3x + y) + f(x + 3y) = 24f(x + y) - 6f(x - y) + 64 f(x) + 64f(y)$ in Paranormed Spaces.*
Keywords : *Hyers-Ulam stability , Quartic Functional Equation, Paranormed Spaces***.**

I. Introduction

In 1940, the stability of functional equations seem to have been first triggered by Stanislaw M. Ulam [21]. He raised the following problem : Given conditions in order for a linear mapping near an approximately additive mapping to exist (see [22]). In 1941, Hyers [8] solved the above problem for the case where G_1, G_2 are Banach spaces. Hyers [8] proved the stability on Cauchy functional equation. Further, this terminology is also applied to another functional equations. In 1978, Th. M. Rassias [16] generalized the result of Hyers theorem by considering the unbounded Cauchy difference :

$\| f(x + y) - f(x) - f(y) \| \le \epsilon (\|x\|^p + \|y\|^p)$ where $\epsilon > 0$ and $p \in [0, 1)$.

In 1994, a further generalization was obtained by Gavruta [7]. Later on, Cadariv and Radu [1] introduced the another approach instead of direct method for solving the stability of functional equations (see [2]) via fixed point theory. During the last few decades, a number of papers and research monographs have been published on various generalization and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [4-6], [10-14],[17-19]). We also refer the readers to books : Czerwik [3] and Hyers et. al. [9].

The functional equation

$$f(3x + y) + f(x + 3y) = 24f(x + y) - 6f(x - y) + 64 f(x) + 64f(y) \qquad (1.1)$$

is said to be quartic functional equation since $f(x) = cx^4$ is the solution of above equation. Every solution of quartic equation is said to be quartic mapping. Monta-Karn Petapirak and Pasian Nakmahachalasint [15] proved the stability problem of above quartic functional equation.

Throughout this paper, we prove the Hyers-Ulam-Rassias of equation (1.1) in paranormed space.

II. Preliminaries

Before giving the main result, we present here some basic facts related to paranormed spaces and some preliminary results. We assume (X, P) is a Frechet space and $(Y, \|.\|)$ is a Banach space.

Definition 2.1 [12] : A normed space over K a pair $(V, \|.\|)$, where V is a vector space over K and $\|.\| : V \to R^+$ such that

(i) $\| x \| = 0$ iff $x = 0$
(ii) $\| \lambda x \| = | \lambda | \| x \|$ for all $\lambda \in K$ and $v \in V$.
(iii) $\| x + y \| \le \| x \| + \| y \|$ for all $x, y \in V$.

Definition 2.2 [20] : Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function such that

(i) $P(0) = 0$;
(ii) $P(-x) = P(x)$;
(iii) $P(x + y) \le P(x) + P(y)$ (Triangle inequality).
(iv) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_0 - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a paranormed spaces if P is a paranorm on X.
The paranorm is called total if, in addition, we obtain
(v) $P(x) = 0$ implies $x = 0$

Definition 2.3 [12] : A Frechet space is a total and complete paranormed spaces.

III. Main Result

In the following theorems, we will prove the stability of quartic functional equation (1.1) in paranormed spaces.

Theorem 3.1 : Let q, θ be positive real numbers with $q > 4$ and suppose $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and

$$P(f(3x + y) + f(x + 3y) - 24f(x + y) + 6f(x + y) - 64f(x) - 64f(y)) \le \theta (\| x \|^q + \| y \|^q) \quad (3.1)$$

for all x, y $\in$ Y. Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that

$$P(f(x) - Q_4(x)) \le \frac{1}{3^q - 81}\theta \| x \|^q \quad (3.2)$$

for all x $\in$ Y.

Proof : Putting y = 0 in (3.1), we get

$$P(f(3x) - 81f(x)) \le \theta \|x\|^q$$

for all x $\in$ Y. So

$$P\left(f(x) - 16f\left(\frac{x}{3}\right)\right) \le \theta \left\|\frac{x}{3}\right\|^q \le \frac{1}{3^q}\theta \| x \|^q$$

$$P\left(f\left(\frac{x}{3}\right) - 16f\left(\frac{x}{3^2}\right)\right) \le \frac{1}{3^q}\theta \left\|\frac{x}{3}\right\|^q \le \frac{1}{3^q.3^q}\theta \| x \|^q$$

for all x, y $\in$ Y. Hence

$$P\left(81^l f\left(\frac{x}{3^l}\right) - 81^m f\left(\frac{x}{3^m}\right)\right) \le \sum_{j=1}^{m-1} P\left(81^j f\left(\frac{x}{3^j}\right) - 81^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right) \le \frac{1}{3^q}\sum_{j=1}^{m-1}\frac{81^j}{3^{qj}}\theta \| x \|^q \quad (3.3)$$

for all non-negative integers m and l with $m > l$ and for all x $\in$ Y. Now, we obtain from (3.3) that the sequence $\left\{81^n f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence for all x $\in$ Y. Because X is complete, the sequence $\left\{81^n f\left(\frac{x}{3^n}\right)\right\}$ converges. So we can define the mapping $Q_4 : Y \to X$ by

$$Q_4(x) = \lim_{n\to\infty} 81^n f\left(\frac{x}{3^n}\right) \quad (3.4)$$

for all x $\in$ Y.

Further, assuming $l = 0$ and passing the limit $m \to \infty$ in (3.3.), we have (3.2). If follows from (3.1) that

$$P(Q_4(3x + y) + Q_4(x + 3y) - 24Q_4(x + y) + 6Q_4(x - y) - 64Q_4(x) - 64Q_4(y))$$

$$= P\left(81^n\left(f\left(\frac{3x+y}{3^n}\right) + f\left(\frac{x+3y}{3^n}\right) - 24f\left(\frac{x+y}{3^n}\right) + 6f\left(\frac{x-y}{3^n}\right) - 64f\left(\frac{x}{3^n}\right) - f\left(\frac{y}{3^n}\right)\right)\right)$$

$$\le \lim_{n\to\infty} 81^n P\left(f\left(\frac{3x+y}{3^n}\right) + f\left(\frac{x+3y}{3^n}\right) - 24f\left(\frac{x+y}{3^n}\right) + 6f\left(\frac{x-y}{3^n}\right) - 64f\left(\frac{x}{3^n}\right) - 64f\left(\frac{y}{3^n}\right)\right)$$

$$\le \lim_{n\to\infty} 81^n \theta\left(\left\|\frac{x}{3^n}\right\|^q + \left\|\frac{y}{3^n}\right\|^q\right)$$

$$\le \lim_{n\to\infty} \frac{81^n}{3^{nq}}\theta\left(\|x\|^q + \|y\|^q\right)$$

$$= 0 \quad \text{for all } x, y \in Y.$$

Thus $Q_4(3x + y) + Q_4(x + 3y) = 24Q_4(x + y) - 6Q_4(x - y) + 64Q_4(x) + 64Q_4(y)$ for all $x, y \in Y$ and so the mapping $Q_4 : Y \to X$ is quartic. Further , suppose $T_4 : Y \to X$ be another quartic mapping satisfying (3.2). Then , we obtain

$$P(Q_4(x) - T_4(x)) = P\left(81^n\left(Q_4\left(\frac{x}{3^n}\right) - T_4\left(\frac{x}{3^n}\right)\right)\right)$$

$$\leq 81^n P\left(Q_4\left(\frac{x}{3^n}\right) - T_4\left(\frac{y}{3^n}\right)\right)$$

$$\leq 81^n P\left(Q_4\left(\frac{x}{3^n}\right) - T_4\left(\frac{x}{3^n}\right) + f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right)$$

$$\leq 81^n\left(P\left(Q_4\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right) + P\left(T_4\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right)\right)$$

$$\leq \frac{81^n.2}{(3^q.81)3^{nq}}\theta\,||\,x\,||^q$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $Q_4(x) = T_4(x)$ for all $x \in Y$. Hence, the uniqueness of Q_4 has been proved. SO, the mapping $Q_4 : Y \to X$ is a unique quartic mapping satisfying (3.2).

Theorem 3.2 : Let q be a real positive number q < 4, and suppose $f : X \to Y$ be a mapping satisfying f(0) = 0 and

$$\|f(3x + y) + f(x + 3y) - 24f(x + y) + 6f(x - y) - 64f(x) - 64f(y)\| \leq P(x)^q + P(y)^q \tag{3.5}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q_4 : X \to Y$ such that

$$||\,f(x) - J_4(x)\,|| \leq \frac{1}{81 - 3^q}P(x)^q \tag{3.6}$$

for all $x \in X$.

Proof : Assuming y = 0 in (3.5), we obtain

$$\|\,f(3x) - 81\,f(x)\,\| \leq P(x)^q$$

$$\|\,f(x) - \frac{1}{81}f(3x)\,\| \leq \frac{1}{81}P(x)^q$$

for all $x \in X$. Similarly,

$$\|\,f(3x) - \frac{1}{81}f(3^2x)\,\| \leq \frac{3^q}{81}P(x)^q$$

for all $x \in X$. Hence

$$\left\|\frac{1}{81^l}f(3^l x) - \frac{1}{81^m}f(3^m x)\right\| \leq \sum_{j=1}^{m-1}\left\|\frac{1}{81^j}f(3^j x) - \frac{1}{81^{j+1}}f(3^{j+1}x)\right\|$$

$$\leq \frac{1}{81}\sum_{j=1}^{m-1}\frac{3^{qj}}{81^j}P(x)^q \tag{3.7}$$

for all non-negative integers m and l with $m > l$ and for all $x \in X$. It follows from (3.7) that the sequence $\left\{\frac{1}{81^n}f(3^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{81^n}f(3^n x)\right\}$ converges. So, we can define the mapping $J_4 : X \to Y$ by $J_4(x) = \lim_{n\to\infty}\frac{1}{81^n}f(3^n x)$ for all $x \in X$.

Further, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.6).

It follows from (3.5) that

$\|\,J_4(3x + y)\,J_4(x + 3y) - 24J_4(x + y) + 6J_4(x - y) - 64J_4(x) - 64J_4(y)\,\|$

$$= \lim_{n\to\infty}\left\|\frac{1}{81^n}f\left(3^n(3x+y)\right) + \frac{1}{81^n}f\left(3^n(x+3y)\right) - \frac{24}{81^n}f\left(3^n(x+y)\right) + \frac{6}{81^n}f\left((3^n(x-y)\right) - \frac{64}{81^n}f\left(3^n(x)\right) - \frac{64}{81^n}f\left(3^n(y)\right)\right\|$$

$$\leq \lim_{n\to\infty}\frac{1}{81^n}\left\|f\left(3^n(3x+y)\right) + f\left(3^n(x+3y)\right) - 24f\left(3^n(x+y)\right) + 6f\left(3^n(x-y)\right) - 64f\left(3^n(x)\right) - 64f\left(3^n(y)\right)\right\|$$

$$\leq \lim_{n\to\infty}\frac{3^{nq}}{81^n}\left(P(x)^q + P(y)^q\right)$$

$$= 0$$

for all $x, y \in X$. Thus $J_4(3x + y)\,J_4(x + 3y) = 24J_4(x + y) - 6J_4(x - y) + 64J_4(x) + 64J_4(y)$ for all $x, y \in X$ and so the mapping $J_4 : X \to Y$ is quartic. Let us suppose $C : X \to Y$ be another quartic mapping satisfying (3.6). Then we obtain

$$\left\|J_4(x) - C(x)\right\| = \frac{1}{81^n}\left\|J_4(3^n x) - C(3^n x)\right\|$$

$$\leq \frac{1}{81^n}\left(\left\|J_4(3^n x) - f(3^n x)\right\| + \left\|C(3^n x) - f(3^n x)\right\|\right)$$

$$\leq \frac{2.3^{nq}}{\left(81 - 3^r\right)81^n}.P(x)^q$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $J_4(x) = C(x)$ for all $x \in X$. This proves the uniqueness of J_4. Hence the mapping $J_4 : X \to Y$ is unique quartic mapping satisfying (3.6).

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