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Exact solutions of N-dimensional Bianchi Type-V space-time in *f* (*R*) theory of gravity

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Abstract: In the present paper, we have obtained two exact vacuum solutions of the field equations in $f(R)$ theory of gravity in *N*-dimensional Bianchi type-V space time which is the generalization of the work of M. Sharif and M. Farasat Shamir (2009) in V_4 and our earlier work V_5 (V_6) too. The first vacuum solution is singular and second one is non-singular for $r \neq 0$ and $r = 0$ respectively. The value of $f(R)$ is also evaluated for both the solutions. Some physical properties of these solutions have been studied. It is interesting to note that all the work in V_4 and V_5 (V_6) can be reproduced by reducing the dimensions.

Keywords: $f(R)$ theory of gravity, *N*-dimensional Bianchi type-I space-time, *N*- dimensional vacuum field equations in $f(R)$ theory of gravity.

I. INTRODUCTION

In the paper [1], M. Sharif and M. Farasat Shamir (2009) have studied the accelerating expansion of the universe using four dimensional Bianchi type-V space time and obtained exact vacuum solutions of four dimensional Bianchi type-V space time in $f(R)$ theory of gravity using metric approach. We have extended the work of M. Sharif and M. Farasat Shamir (2009) to higher five and six dimensional Bianchi type-V space time in our earlier papers [2] and [3] respectively and obtained similar exact vacuum solutions in V_5 (V_6). It has been observed that the work regarding exact vacuum solutions of Einstein's modified theory of gravity in V_6 can further be extended to higher *N*-dimensional space-time to study the accelerating expansion of the universe using the vacuum field equations in $f(R)$ theory of gravity and therefore an attempt has been made in the present paper. Thus, in this paper, we propose to solve the Einstein's modified field equations of $f(R)$ theory of gravity using the metric approach in *N*-dimensional Bianchi type-V space-time to investigate different two models of the universe.

The paper is organized as follows : In section 2, we give a brief introduction about the field equations in $f(R)$ theory of gravity in V_n . Sections-3 is used to find exact vacuum solutions of *N*-dimensional Bianchi type-V space-time and made the analysis of singularity of these solutions. Section-4 and 5 are dealt with *N*-dimensional models of the universe for $r \neq 0$ and $r = 0$ respectively and in the last section-6, we summarize and conclude the results.

§ 2. Field equations in $f(R)$ theory of gravity in V_n

The $f(R)$ theory of gravity is nothing but the modification of general theory of relativity proposed by Einstein. In the $f(R)$ theory of gravity there are two approaches to find out the solutions of modified Einstein's field equations. Thus in the present paper, we propose to solve the Einstein's modified field equations of $f(R)$ theory of gravity using the metric approach with the help of the vacuum field equations in *N*-dimensional Bianchi type-V space-time. The corresponding field equations of $f(R)$ gravity theory in V_n are given by

$$F(R)R_{ij} - \frac{1}{2} f(R)g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = kT_{ij}, \quad (i, j = 1,2,\cdots,n) \qquad (1)$$

where $F(R) \equiv \dfrac{df(R)}{dR}, \qquad \Box \equiv \nabla^i \nabla_i \qquad (2)$

with ∇_i the covariant derivative and T_{ij} is the standard matter energy momentum tensor. These are the fourth order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left hand side of the equation. If we consider $f(R) = R$, these equations of $f(R)$ theory of gravity reduce to the field equations of Einstein's general theory of relativity in V_n.

After contraction of the field equation (1), we get

$$F(R)R - \frac{n}{2} f(R) + (n-1)\Box F(R) = kT \quad . \tag{3}$$

In vacuum this field equation (3) reduces to

$$F(R)R - \frac{n}{2} f(R) + (n-1)\Box F(R) = 0. \tag{4}$$

This yields a relationship between $f(R)$ and $F(R)$ which can be used to simplify the field equations and to evaluate $f(R)$.

§ 3. Exact Vacuum Solutions of *N*- dimensional Bianchi type-V Space-Time

In this section we propose to find exact vacuum solutions of *N*- dimensional Bianchi type-V space-time in $f(R)$ gravity.

The line element of the *N*-dimensional Bianchi type-V space-time is given by

$$ds^2 = dt^2 - A^2(t)dx_1^2 - e^{2mx_1}[B^2(t)dx_2^2 + C^2(t)\sum_{i=3}^{n-1} dx_i^2] \tag{5}$$

where A, B and C are cosmic scale factors and m is an arbitrary constant. The corresponding Ricci scalar is

$$R = -2[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + (n-3)\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + (n-3)\frac{\dot{B}\dot{C}}{BC} + (n-3)\frac{\dot{A}\dot{C}}{AC} + \frac{(n-3)(n-4)}{2}\frac{\dot{C}^2}{C^2} - \frac{(n-1)(n-2)}{2}\frac{m^2}{A^2}], \tag{6}$$

Where dot denotes the derivative with respect to t.

We define the average scale factor a as

$$a = (ABC^{n-3})^{\frac{1}{n-1}}, \tag{7}$$

and the volume scale factor is define as

$$V = a^{n-1} = ABC^{n-3}. \tag{8}$$

The generalized mean Hubble parameter H is defined by

$$H = \frac{1}{n-1}\sum_{i=1}^{n-1} H_i, \tag{9}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$, $H_3 = H_4 = \ldots = H_{n-1} = \frac{\dot{C}}{C}$ are the directional Hubble parameters in the directions of $x_1, x_2, x_3, \ldots, x_{n-1}$ axes respectively. Using equations (7), (8) and (9) , we obtain

$$H = \frac{1}{(n-1)}\frac{\dot{V}}{V} = \frac{1}{(n-1)}\sum_{i=1}^{n-1} H_i = \frac{\dot{a}}{a}. \tag{10}$$

From equation (4) we have

$$f(R) = \frac{2}{n}[(n-1)\Box F(R) + F(R)R]. \tag{11}$$

Putting this value of $f(R)$ in the vacuum field equations (4), we obtain

$$\frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} = \frac{1}{n}[F(R)R - \Box F(R)]. \tag{12}$$

Since the metric (5) depends only on *t*, one can view equation (12) as the set of differential equations for F(t), A, B and C. It follows from equation (12) that the combination

$$A_i = \frac{F(R)R_{ii} - \nabla_i \nabla_i F(R)}{g_{ii}}, \tag{13}$$

is independent of the index i and hence $A_i - A_j = 0$ for all i and j. Consequently $A_0 - A_1 = 0$ gives

$$-\frac{\ddot{B}}{B} - (n-3)\frac{\ddot{C}}{C} - (n-2)\frac{m^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + (n-3)\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0, \tag{14}$$

Also, $A_0 - A_2 = 0$,

$$-\frac{\ddot{A}}{A} - (n-3)\frac{\ddot{C}}{C} - (n-2)\frac{m^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + (n-3)\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{B}\dot{F}}{BF} - \frac{\ddot{F}}{F} = 0, \tag{15}$$

Similarly $A_0 - A_3 = 0$, $A_0 - A_4 = 0$, …, $A_0 - A_{n-1} = 0$ give respectively

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - (n-4)\frac{\ddot{C}}{C} - (n-2)\frac{m^2}{A^2} + (n-4)\frac{\dot{C}^2}{C^2} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{F}}{CF} - \frac{\ddot{F}}{F} = 0, \tag{16}$$

The 01-component can be written by using equation (1) in the following form

$$\frac{(n-2)\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{(n-3)\dot{C}}{C} = 0 \tag{17}$$

Subtracting equation (15), (16) and (16) from equation (14), (15) and (14), we get

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + (n-3)\frac{\dot{C}}{C}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) + \frac{\dot{F}}{F}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) = 0, \tag{18}$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{(n-4)\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{(n-4)\dot{B}\dot{C}}{BC}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{F}}{F}(\frac{\dot{B}}{B}-\frac{\dot{C}}{C})=0\,, \tag{19}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-(n-4)\frac{\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+(n-4)\frac{\dot{A}\dot{C}}{AC}-\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{F}}{F}(\frac{\dot{A}}{A}-\frac{\dot{C}}{C})=0\,. \tag{20}$$

After integration the above equations imply that

$$\frac{B}{A}=d_1\exp[\ c_1\int\frac{dt}{a^{n-1}F}]\,, \tag{21}$$

$$\frac{C}{B}=d_2\exp[\ c_2\int\frac{dt}{a^{n-1}F}]\,, \tag{22}$$

$$\frac{A}{C}=d_3\exp[\ c_3\int\frac{dt}{a^{n-1}F}] \tag{23}$$

where c_1, c_2, c_3 and d_1, d_2, d_3 are constants of integration which satisfied the relation $c_1+c_2+c_3=0$ and $d_1\,d_2 d_3=0$.

From equations (21) , (22), (23) the metric functions are obtained explicitly as

$$A=ap_1\exp[\ q_1\int\frac{dt}{a^5F}]\,, \tag{24}$$

$$B=ap_2\exp[\ q_2\int\frac{dt}{a^5F}]\,, \tag{25}$$

$$C=ap_3\exp[\ q_3\int\frac{dt}{a^5F}] \tag{26}$$

where $p_1=[d_1^{-(n-2)}d_2^{-(n-3)}]^{1/(n-1)}$, $p_2=(d_1 d_2^{-(n-3)})^{1/(n-1)}$, $p_3=(d_1 d_2^{2})^{1/(n-1)}$ (27)

and $q_1=-\dfrac{(n-2)c_1+(n-3)c_2}{n-1}$, $q_2=\dfrac{c_1-(n-3)c_2}{n-1}$, $q_3=\dfrac{c_1+2c_2}{n-1}$. (28)

We have pointed out that p_1, p_2, p_3 and q_1, q_2, q_3 are related by

$$p_1 p_2 p_3^{(n-3)}=1\,, q_1+q_2+(n-3)q_3=0\,. \tag{29}$$

From equations (24), (25) and (26), the metric functions becomes

$$A=a\,, \tag{30}$$

$$B=ap\exp[Q\int\frac{dt}{a^{n-1}F}]\,, \tag{31}$$

$$C=ap^{-1/n-3}\exp[-\frac{Q}{n-3}\int\frac{dt}{a^{n-1}F}] \tag{32}$$

where, $p_1=1$, $p_2=p_3=p$, (33)

and

$$q_1 = 0 , \quad q_2 = (n-3) q_3 = Q . \tag{34}$$

Now we use the power law assumption to solve the integral part in the above equations as

$$F \; \alpha \; a^m$$

where m is an arbitrary constant.

$$F = k a^m , \tag{35}$$

where k is the constant of proportionality and m is any integer.

The deceleration parameter q in cosmology is the measure of the cosmic acceleration of the universe expansion and is defined as

$$q = -\frac{a \ddot{a}}{\dot{a}^2} . \tag{36}$$

It has been pointed out that initially, q was supposed to be positive but recent observations from the supernova experiments suggest that q is negative. Thus the sign of q plays an important role to identify the behaviour of the universe. The positive deceleration parameter corresponds to a decelerating model while the negative value provides inflation.

The well-known relation between the average Hubble parameter H and average scale factor a given as

$$H = l a^{-r} , \tag{37}$$

where $l > 0$ and $r \geq 0$.

From equation (10) and (37), we have

$$\dot{a} = l a^{1-r} \tag{38}$$

and consequently the deceleration parameter becomes

$$q = r - 1 \tag{39}$$

which is a constant. After integrating equation (38), we have

$$a = (rlt + k_1)^{1/n} , \qquad r \neq 0 \tag{40}$$

and

$$a = k_2 \exp(\; lt) , \qquad r = 0 , \tag{41}$$

k_1 and k_2 are constants of integration.

Thus we have two values of the average scale factors which correspond to two different models of the universe.

From the scalar R given in the equation (6), we can check the singularity of the solutions. If we consider $m = -(n-2)$ as a special case then from equation (35) we have

$$F = k a^{-(n-2)} . \tag{42}$$

After some manipulations, we can write

$$R_1 = -\frac{2}{a^2k^2}\{(n-1)k^2(\ddot{a}\,a + \frac{(n-2)}{2}\dot{a}^2) + \frac{n-2}{2(n-3)}Q^2 - \frac{(n-1)(n-2)}{2}k^2m^2\} \qquad (43)$$

which shows that singularity occurs at $a = 0$.

§ 4. *N*- dimensional Model of the Universe when $r \neq 0$

In this section we study the *N*- dimensional model of the universe for $r \neq 0$.

When $r \neq 0$ then we have $a = (rlt + k_1)^{1/n}$ and for $m = -(n-2)$ as a special case, F becomes

$$F = k(rlt + k_1)^{-(n-2)/r} \qquad (44)$$

For this value of F , equations (30), (31) and (32), imply that

$$A = (rlt + k_1)^{1/r}\ , \qquad (45)$$

$$B = p(rlt + k_1)^{1/r}\exp[\frac{Q(rlt + k_1)^{\frac{r-1}{r}}}{kl(n-1)}], \qquad r \neq 1 \qquad (46)$$

$$C = p^{-1/n-3}(rlt + k_1)^{1/r}\exp[-\frac{Q(rlt + k_1)^{\frac{r-1}{r}}}{(n-3)kl(r-1)}], \qquad r \neq 1\ . \qquad (47)$$

The directional Hubble parameters $H_i\,(i = 1,2,3,\cdots,n-1)$ take the form

$$H_i = \frac{l}{nlt + k_1} + \frac{q_i}{k(nlt + k_1)^{1/n}}\ . \qquad (48)$$

The mean generalized Hubble parameter becomes

$$H = \frac{l}{rlt + k_1} \qquad (49)$$

And the volume scale factor becomes

$$V = (rlt + k_1)^{n-1/r}\ . \qquad (50)$$

From the equation (11), the function $f(R)$, found as

$$f(R) = \frac{2k}{n}(rlt + k_1)^{-\frac{(n-2)}{r}}R + \frac{2(n-1)(n-2)}{n}kl^2(r-1)(rlt + k_1)^{\frac{-2r-(n-2)}{r}}\ . \qquad (51)$$

From equation (43) Ricci scalar $R = R_1$ becomes

$$R = R_1 = -2\{l^2(n-1)(\frac{n}{2} - r)(rlt + k_1)^{-2} + \frac{(n-2)}{2(n-3)}[Q^2 - (n-1)(n-3)k^2m^2](rlt + k_1)^{-2/r}\} \quad (52)$$

which clearly indicates that $f(R)$ cannot be explicitly written in terms of R. However, by inserting this value of R, $f(R)$ can be written as a function of t, which is true as R depends upon t. For a special case when $n = 1/2$, $f(R)$ turns out to be

$$f(R) = \frac{2k}{n}\left[\frac{-(n-1)^2l^2 \pm \sqrt{(n-1)^4l^4 - \frac{4(n-2)}{(n-3)k^2}[Q^2 - (n-1)(n-3)k^2m^2]R}}{2R}\right]^{-(n-2)} R$$

$$-\frac{(n-1)(n-2)kl^2}{n}\left[\frac{-(n-1)^2l^2 \pm \sqrt{(n-1)^4l^4 - \frac{4(n-2)}{(n-3)k^2}[Q^2 - (n-1)(n-3)k^2m^2]R}}{2R}\right]^{-(n-1)} \quad (53)$$

This gives $f(R)$ only as a function of R.

§ 5. *N*- dimensional Model of the Universe when $r = 0$

In this section, we study the *N*- dimensional model of the universe for $r = 0$.
For $r = 0$ the average scale factor for the model of the universe is $a = k_2 \exp(lt)$ and hence F becomes

$$F = \frac{k}{k_2^{(n-2)}} \exp[-(n-2)lt]. \quad (54)$$

For this value of F equations (30), (31) and (32), imply that

$$A = k_2 \exp(lt), \quad (55)$$

$$B = pk_2 \exp(lt) \exp[-\frac{Q\exp(-lt)}{klk_2}], \quad (56)$$

$$C = p^{-1/n-3} k_2 \exp(lt) \exp[\frac{Q\exp(-lt)}{(n-3)klk_2}]. \quad (57)$$

The mean generalized Hubble parameter becomes

$$H = l. \quad (58)$$

And the volume scale factor becomes

$$V = k_2^{(n-2)} \exp[(n-1)lt] . \tag{59}$$

From the equation (11), the function $f(R)$, found as

$$f(R) = \frac{2k}{nk_2^{(n-2)}} \exp[-(n-2)lt][R-(n-1)(n-2)l^2] . \tag{60}$$

From equation (43) Ricci scalar $R = R_1$ becomes

$$R = R_1 = -2\{\frac{n(n-1)}{2} l^2 + \frac{(n-2)}{2(n-3)k^2 k_2^2 \exp(2lt)} [Q^2 - (n-1)(n-3)k^2 m^2]\} . \tag{61}$$

The general function $f(R)$ in terms of R

$$f(R) = \frac{2k^{n-1}}{n} \left\{ \frac{(n-3)[R+n(n-1)l^2]}{(n-2)[(n-1)(n-3)k^2 m^2 - Q^2]} \right\}^{n-2/2} [R-(n-1)(n-2)l^2] , \tag{62}$$

which correspond the general function $f(R)$

$$f(R) = \sum a_r R^r , \tag{63}$$

where r may take the values from negative or positive.

§ 6. Concluding Remark

Recently, in the paper [1], M. Sharif and M. Farasat Shamir (2009) have investigated two exact vacuum solutions of the four dimensional Bianchi type -V space time in $f(R)$ theory of gravity by using the variation law of Hubble parameter to discuss the well-known phenomenon of the universe expansion. This work regarding exact vacuum solutions of Einstein's modified theory of gravity has further been extended to higher five(six)-dimensional Bianchi type-V space-times in our paper refer them to [2] and [3] respectively.

In this paper, we have extended the study regarding exact vacuum solutions in $f(R)$ theory of gravity to *N*-dimensional Bianchi type -V space time and obtained two exact vacuum solutions corresponding to two models of the universe (i.e., $r \neq 0$ and $r = 0$).The first solution gives a singular model with power law expansion and positive deceleration parameter while the second solution gives a non-singular model with exponential expansion and negative deceleration parameter. The functions $f(R)$ are evaluated for both models.

The physical behavior of these *N*-dimensional models is observed as under :

i. For $r \neq 0$ i.e. singular *N*-dimensional model of the universe

For this model average scale factor $a = (rlt + k_1)^{1/r}$.

This model has point singularity at $t \equiv -k_1 / rl$.

The physical parameters $H_1, H_2, H_3, H_4, \ldots . H_{n-1}$ and H are all infinite at this point

The volume scale factor V vanishes at this point.

The function of the Ricci scalar, $f(R)$, is also infinite.

The metric functions A, B and C vanish at this point of singularity.

In this way we can conclude from these observations that the N- dimensional model of the universe starts its expansion with zero volume at $t = -k_1 / rl$ *and it continues to expand.*

ii. For $r = 0$ **i.e. non-singular *N*-dimensional model of the universe**

For this model average scale factor $a = k_2 \exp(lt)$.

This *N*-dimensional model of the universe is non-singular because exponential function is never zero and hence there does not exist any physical singularity for this *N*-dimensional model of the universe.

The physical parameters $H_1, H_2, H_3, H_4, \ldots.\ H_{n-1}$ are all finite for all finite values of t.

The mean generalized Hubble parameter H is constant.

The function of the Ricci scalar, $f(R)$ is also finite.

The metric functions A, B and C do not vanish for this model.

The volume scale factor increases exponentially with time which indicates that the N *-dimensional model of the universe starts its expansion with zero volume from infinite past.*

The solution obtained here is more general to that of earlier exact vacuum solutions obtained in $V_4, V_5\ (V_6)$. It is shown that results regarding accelerating expansion of the universe obtained in *N*- dimensional Bianchi type-V space time are similar to those of V_4, V_5 and V_6, essentially retaining their mathematical format.

It is pointed out that the work of M. Sharif and Farasat Shamir (2009) in V_4 and our work in $V_5\ (V_6)$ regarding accelerating expansion of the universe emerge as special cases of our work carried out in the present paper.

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