Research Inventy: International Journal Of Engineering And Science
Vol.3, Issue 5 (July 2013), Pp 54-57
Issn(e): 2278-4721, Issn(p):2319-6483, Www.Researchinventy.Com

On The Non Homogeneous Ternary Quintic Equation

$$x^2 - xy + y^2 = 7z^5$$

M.A.Gopalan, S. Vidhyalakshmi, A.Kavitha, M.Manjula
Department of Mathematics,Shrimati Indira Gandhi College,Tiruchirappalli – 620 002.

ABSTRACT: *The ternary Quintic Diophantine Equation given by is* $x^2 - xy + y^2 = 7z^5$ *analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*

KEY WORDS: *Quintic equation with three unknowns, Integral solutions.*

I. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since ambiguity[1-3].For illustration, one may refer [4-9] for quintic equation with three unknowns . This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous quintic equation with three unknowns given by $x^2 - xy + y^2 = 7z^5$.A few relations among the solutions are presented.

II. NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m .
- P_n^m - Pyramidal number of rank n with size m .
- $ct_{m,n}$ - Centered polygonal number of rank n with size m .
- gn_a - Gnomonic number of rank a
- so_n - Stella octangular number of rank n
- s_n - Star number of rank n
- pr_n - Pronic number of rank n
- pt_n - Pentatope number of rank n
- $CP_{m,n}$ - Centered pyramidal number of rank n with size m .
- $f_{m,s}^n$ - m-dimensional figurate number of rank n with s sides.

III. METHOD OF ANALYSIS

The quintic Diophantine equation with three unknowns to be solved for its non-zero distinct integral solution is

$$x^2 - xy + y^2 = 7z^5 \quad (1)$$

Introduction of the linear transformation

$$x = u + v \text{ and } y = u - v \quad (2)$$

in (1) leads to $u^2 + 3v^2 = 7z^5$ (3)

Different patterns of solutions of (3) and hence that of (1) using (2) are given below

1. Pattern -1

Let $z = z(a,b) = a^2 + 3b^2$ (4)

Write $7 = (2 + i\sqrt{3})(2 - i\sqrt{3})$ (5)

Using (4) & (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{3}v = (2 + i\sqrt{3})(a + i\sqrt{3}b)^5$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = 2a^5 - 15a^4b - 60a^3b^2 + 90a^2b^3 + 90ab^4 - 27b^5$$

$$v = v(a,b) = a^5 + 10a^4b - 30a^3b^2 - 60a^2b^3 + 45ab^4 + 18b^5$$

Substituting the above values of u and v in (2), we get

$$x = x(a,b) = 3a^5 - 5a^4b - 90a^3b^2 + 30a^2b^3 + 135ab^4 - 9b^5 \quad (6)$$

$$y = y(a,b) = a^5 - 25a^4b - 30a^3b^2 + 150a^2b^3 + 45ab^4 - 45b^5 \quad (7)$$

Thus (4), (6) and (7) represent the non-zero distinct solutions of (1) in two parameters.
A few interesting properties observed are as follows:

1. $x(a,1) + y(a,1) + 54 = 120\ f^{a}_{5,6} - 130\ f^{a}_{4,6} + 15\ CP_{7,a} + 344\ t_{3,a} + 53\ t_{4,a}$
2. $y(a,1) - z^2(a,1) - 224 = 120\ f^{a}_{5,3} - 30\ f^{a}_{4,6} - 25\ SO_a + 48\ S_a + 668\ t_{3,a} - 332\ t_{4,a}$
3. $x(a,1) - 3y(a,1) - 126 = 300\ f^{a}_{4,4} - 150\ OH_a - 545\ t_{4,a}$
4. $x(a,1) - 3y(a,1) + 39\ t_{4,3a} = 10\ S_{a^2} + t_{22,a^2} + 116$
5. $x(a,1) - 3y(a,1) + 41z(a,1) = 35\ t_{6,a^2} - 43\ gn_{(2a)^2} + 206$
6. $x(a,1) + y(a,1) \equiv 0 \pmod 2$
7. $x(a,1) - 3y(a,1) \equiv 56 \pmod{70}$
8. $3y(1,b) - x(1,b) \equiv 0 \pmod b$
9. $38416\ \{5x(a,1) - y(a,1) - 420\ CP_{6,a}\}$ is a quintic integer.

2. Pattern-2:

In addition to (5),

Write 7 as $7 = \frac{1}{4}(1 + i3\sqrt{3})(1 - i3\sqrt{3})$

Following the procedure similar to Pattern-1, the non-zero distinct integer values of x, y and z satisfying (1) are given by

$$x = x(a,b) = 2a^5 - 20a^4b - 60a^3b^2 + 120a^2b^3 + 90ab^4 - 36b^5$$

$$y = y(a,b) = -a^5 - 25a^4b + 30a^3b^2 + 150a^2b^3 - 45ab^4 - 45b^5$$

$$z = z(a,b) = a^2 + 3b^2$$

Properties:

1. $x(a,1) - y(a,1) - 9 = 120\ f^{a}_{5,5} - 90\ f^{a}_{4,6} - 45\ SO_a + 78\ \Pr_a - 118\ t_{4,a}$
2. $x(a,1) + 2y(a,1) + 126 = 395\ t_{4,a} + 150\ Pa^5 - 150\ f^{a}_{4,6}$
3. $x(a(a+1),1) + 2y(a(a+1),1) + 140\ (t_{3,a})^4 + 126 = 420\ (\Pr_a)^2$
4. $x(1,b) + 2y(1,b) + 35\ CP_{6,2b} + 63\ (gn_{63b} - 1) = 70\ SO_b$
5. $x(a,1) - y(a,1) + z^2(a,1) \equiv 0 \pmod 3$

3. Pattern-3:

Also, Write 7 as $7 = \frac{1}{4}(5 + i\sqrt{3})(5 - i\sqrt{3})$

For this choice of 7, the non-zero distinct integer value of x, y and z satisfying (1) are given by

$$x = x(a,b) = 3a^5 + 5a^4b - 90a^3b^2 - 30a^2b^3 + 135ab^4 + 9b^5$$

$$y = y(a,b) = 2a^5 - 20a^4b - 60a^3b^2 + 120a^2b^3 + 90ab^4 - 36b^5$$

$$z = z(a,b) = a^2 + 3b^2$$

Properties:

1. $x(a,1) - y(a,1) - 45 = 24 f^a_{5,7} - 186 f^a_{4,6} + 104 Pa^5 + 90 t_{3,a} - 191 t_{4,a}$
2. $x(a,1) - 9 = 72 f^a_{5,5} - 90 f^a_{4,6} - 24 CP_{3,a} - 6 Pa^5 - t_{92,a} - 13 t_{4,a^2} + 2t_{9,a} - 7t_{4,a}$
3. $x(a,1) + y(a,1) + 27 = 120 f^a_{5,7} - 540 f^a_{4,4} - 50 CP_{3,a} + 340 \Pr_a - 55 t_{4,a}$
4. $2x(a,1) + 3y(a,1) \equiv 0 \pmod 2$
5. $4x(a,1) + y(a,1) \equiv 14 \pmod{30}$
6. $8\{2x(1,b) - 3y(1,b) + 70 SO_b - 63[gn_{b^5} - 1]\}$ is a cubical integer
7. Each of the following represents a nasty number:
   - $-2\{4x(1,b) + y(1,b) - 14 gn_{22b^4} + 4t_{4,b^2}\}$
   - $6z(a,a)$

IV. REMARKABLE OBSERVATIONS

I: Consider x and y to be the length and breadth of a rectangle R, whose area, perimeter and length of its diagonal are respectively denoted by A,P and L.

Then, it is noted that,

- $P^2 - 12A \equiv 0 \pmod{28}$
- $L^2 - A \equiv 0 \pmod 7$
- $2401\{L^2 - A\}$ is a quintic integer

II: Employing the integral solutions of (1), a few interesting results among the special numbers are exhibited.

1) $2401\left\{\left[\frac{3P^3_x}{t_{3,x+1}}\right]^2 + \left[\frac{6P^3_{y-2}}{\Pr_{y-2}}\right]^2 - \left[\frac{P^4_{x-1}}{t_{3,2x-2}}\right]\left[\frac{4P^5_y}{t_{3,y}}\right]\right\}$ is a quintic integer.

2) Each of the following represents a nasty number:

- $-6\left[\frac{36P^3_{x-2}}{S_{x-1}-1}\right]^2 + 6\left[\frac{4P^5_y}{t_{3,y}}\right]\left[\frac{P^4_x}{t_{6,x+1}}\right] + 42\left[\frac{t_{3,2z-1}}{gn_z}\right]^5$
- $-6\left[\frac{3(P^4_{y-1} - P^3_{y-1})}{t_{3,y-2}}\right] + 6\left[\frac{P^5_x}{t_{3,x}}\right]\left[\frac{12P^5_y}{S_{y-1}-1}\right] + 42\left[\frac{6P^4_{z-1}}{t_{3,2z-2}}\right]^5$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES:

[1]. L.E.Dickson, History of Theory of Numbers, Vol.11, Chelsea Publishing company, New York (1952).
[2]. L.J.Mordell, Diophantine equations, Academic Press, London(1969).
[3]. Carmichael ,R.D.,The theory of numbers and Diophantine Analysis,Dover Publications, New York (1959)
[4]. M.A.Gopalan & A.Vijayashankar, An Interesting Diophantine problem $x^3 - y^3 = 2z^5$ Advances in Mathematics, Scientific Developments and Engineering Application, Narosa Publishing House,2010, Pp 1-6.

[5]. M.A.Gopalan & A.Vijayashankar, Integral solutions of ternary quintic Diophantine quation $x^2 + (2k+1)y^2 = z^5$,International Journal of Mathematical Sciences 19(1-2), (jan-june 2010),165-169.
[6]. M.A.Gopalan,G.Sumathi & S.Vidhyalakshmi, Integral solutions of non-homogeneous quintic equation with three unknowns $x^2 + y^2 - xy + x + y + 1 = (k^2 + 3)^n z^5$, International Journal of Innovativeresearch in Science, Engineering and technology, Vol 2, Issue 4, April 2013, Pp 920-925.
[7]. S.Vidhyalakshmi ,K.Lakshmi and M.A.Gopalan , Integral solutions of non-homogeneous ternary quintic equation $ax^2 + by^2 = (a+b)z^5, a, b > 0$,Archimedes J.math., 3(2),2013,Pp 197-204.
[8]. S.Vidhyalakshmi, K.Lakshmi and M.A.Gopalan , Integral solutions of non-homogeneous ternary quintic equation $ax^2 - by^2 = (a-b)z^5, a, b > 0$,International journal of computational Engineering research, Vol 3, Issue 4, April 2013,Pp 45-50.
[9]. M.A.Gopalan,G.Sumathi & S.Vidhyalakshmi, Integral solutions of non-homogeneous ternary quintic equation in terms of pells sequence $x^3 + y^3 + xy(x+y) = 2z^5$,JAMS, Vol.6, no.1,April 2013, page no.59-62