Research Inventy: International Journal Of Engineering And Science
Issn: 2278-4721, Vol. 2, Issue 4 (February 2013), Pp 36-41
Www.Researchinventy.Com

New Oscillation Criteria For Second Order Nonlinear Differential Equations

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Abstract: *Consider the second order nonlinear differential equations with damping term and oscillation's nature of* $(r(t)x'(t))'+p(t)x'(t)+q(t)f(x(t))k(x'(t))=0 \qquad t \ge t_0$ *(1)*
to used oscillatory solutions of differential equations
$(\alpha(t)x'(t))'+\beta(t)f(x(t))k(x'(t))=0$ *(2)*
where $\alpha(t)$ *and* $\beta(t)$ *satisfy conditions given in this work paper. Our results extend and improve some previous oscillation criteria and cover the cases which are not covered by known results. In this paper, by using the generalized Riccati's technique and positive function H (t, s) of Philo we get a new oscillation and nonoscillation criteria for (1).*

Key Words: *equations, differential, interval, criteria, damping, second order etc.*

I. Introduction

In this paper we are being considered the oscillation solutions in the second order nonlinear functional differential equation:

$$(r(t)x'(t))'+p(t)x'(t)+q(t)f(x(t))k(x'(t))=0, \quad t \ge t_0 \tag{1}$$

where $t, p, q, \in C([t_0,\infty],\Re)$ and $f, g \in (\Re,\Re)$.
In the following we shall assume the conditions

A1) For all $t \in I$, $p(t)>0$, $r(t)>0$, for $t \in I=[\alpha,\infty)$, and $\int_\alpha^\infty \frac{1}{r(t)}dt=\infty$.

A2) $q(t)$ is a real value and locally integrated over *I*.

A3) $xf(x)>0$, and $f'(x) \ge k > 0$.

A4) $k(x'(t)) \ge c_1 > 0$

By the solution of equation (1) or (2) we consider a function $x(t),\ t \in [t_x,\infty) \subset [t_0,\infty\}$ which is twice continuously differentiable and satisfies equation (1) or (2) on the given interval. The number depends on that particular solution x *(t)* under consideration. We consider only non-trivial solutions. A solution *x(t)* of (1) or (2) is said to be oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the interval $[t_0,\infty\}$, so that $\lim_{n\to\infty}\lambda_n=\infty$ and $x(\lambda_n)=0$, $n \in N$, otherwise it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory, otherwise it is considered that is nonoscillatory equation.

The Conditions for oscillatory solutions of the second order differential equations (1) are studied by many authors (see [5], [6], etc.) .Here, we give some conditions for coefficients where the equation (1) has oscillatory solutions and we also take into account the result we have obtained in the previous researches, here we present more generalized criteria that define oscillation solution of the equation (1) to used oscillatory solutions of (2).In this paper are presented theorems, that use generalized Riccati – type transformations, and averaging technique, which explain results for oscillatory nature of differential equations. Also, our results extend and improve a number of existing results (see [5], [9], [10], etc.).

II. Main Result

What follows is, $E(l)$, $\beta(\alpha)$ denote

$$E(l) = e^{\int_{\alpha}^{t} \frac{q(l)}{r(l)} dl}$$

and $$B(\alpha) = \int_{\alpha}^{\infty} (q(l)c_1 - \frac{p^2(l)}{4k^2 r(l)}) dl$$

that we will use in the following theorem.

Theorem 1: The equation (1) is oscillatory if for $p(t) \geq 0, t \geq \alpha$, and

$$\int_{\alpha}^{\infty} \frac{1}{E(l)r(l)} dl = \infty \quad (2)$$

$$B(\alpha) = \infty \quad (3)$$

Proof: For $E(l) = e^{\int_{\alpha}^{t} \frac{q(l)}{r(l)} dl}$, we have $E'(t) = \frac{q(t)}{r(t)} E(t)$, where $E(t) > 0$,

and the equation

$$(E(t)r(t)x'(t))' + E(t)p(t)f(x(t))k(x'(t)) = 0 \quad (4)$$

may be reduced in (1).

We se that (1) is oscillatory if and only if the equation (4) is oscillatory.

Assume that (1) is nonoscillatory. Then there exists a nonoscillatory solution *x (t)* of (1). So we may assume that $x(t) > 0$ on $[t_1, \infty)$, for some $t_1 > \alpha$. We show that $x'(t) > 0$, for $t \geq t_1$.

From (4) we obtain that

$$(E(t)r(t)x'(t))' = -E(t)p(t)x(t)k(x'(t)) \leq 0$$

from where $E(t)r(t)x'(t)$ is not increasing for $t \geq t_1$. Assume that $E(t_2)r(t_2)x'(t_2) < 0$ for some $t_2 > t_1$. Put $E(t_2)r(t_2)x'(t_2) = L$, then for $t \geq t_2$, we have

$$E(t)r(t)x'(t) \leq L .$$

Dividing both sides by $E(t)r(t)$ and integrating from t_2 to t $(> t_2)$, we obtain

$$x(t) - x(t_2) \leq L \int_{t_2}^{t} \frac{1}{E(l)r(l)} dl \ .$$

Because $L \int_{t_2}^{t} \frac{1}{E(l)r(l)} dl$ is tending to $-\infty$, where $t \to \infty$, we conclude that $x(t) < 0$, for sufficiently large t , which is a contradiction. Therefore,

$$x'(t) > 0 \text{, for } t \geq t_1 .$$

In case that $x(t) < 0$, put $y(t) = -x(t)$. So we have $x'(t) > 0$.

Considering the function

$$W(t) = \frac{r(t)x'(t)}{f(x(t))}$$

we have

$$W'(t) = -\frac{p(t)}{r(t)} W(t) - q(t)k(x'(t)) - \frac{W^2(t)f'(x(t))}{r(t)}$$

and from A3), A4) we obtain

$$W'(t) \leq -\frac{p(t)}{r(t)} W(t) - q(t)c_1 - \frac{W^2(t)k}{r(t)} , \quad (5)$$

$$W'(t) \le -\frac{k}{r(t)}(W(t)+\frac{p(t)}{2k})^2+\frac{p^2(t)}{4kr(t)}-q(t)c_1$$

$$W'(t) \le -\frac{k}{r(t)}(W(t)+\frac{p(t)}{2k})^2-(q(t)c_1-\frac{p^2(t)}{4kr(t)})$$

Integrating (6) from t_2 to t $(> t_2)$, we get

$$W(t)-W(t_1)+\int_{t_1}^{t}(q(l)c_1-\frac{p^2(l)}{4k^2r(l)})dl \le -\int_{t_1}^{t}\frac{k}{r(l)}(W(l)-\frac{p(l)}{2k})^2 dl \ .$$

By means of (4) there exists a $t_3 \ge t_1$, such that for $t \ge t_3$, we gain

$$W(t) \le -\int_{t_1}^{t}\frac{k}{r(l)}(W(l)-\frac{q(l)}{2k})^2 dl$$

which is impossible because $W(t) > 0$, for $t \ge t_1$.

To use the theorem above we can prove that for function *W (t)* is truth the following lemma.

Lemma 1: Assume that for $t \ge \alpha$, $p(t) \ge 0$, $q(t) \ge 0$ and (3) are valid. If the differential equations (1) have a positive solution, we have

$$\lim_{t\to\infty}\frac{r(t)x'(t)}{f(x(t))}=0.$$

Proof: Let $x(t) > 0$, be a solution of (1). From Theorem 1 it follows that from $p(t) \ge 0$, for $t \ge \alpha$, and (3) that there exists a $t_1 \ge \alpha$ such that $x'(t) > 0$, for $t \ge t_1$.

Put

$$W(t)=\frac{r(t)x'(t)}{f(x(t))}>0$$

for $t \ge t_1$, and consider Riccati inequation

$$W'(t) \le -\frac{p(t)}{r(t)}W(t)-q(t)c_1-\frac{W^2(t)k}{r(t)}$$

it is obvious that dividing with $W^2 > 0$, we get

$$-\frac{W'(t)}{W(t)} \ge \frac{k}{r(t)}.$$

Integrating the above inequation over $[t_1,\infty)$ and considering the condition $r(t) > 0$, and

$\int_{\alpha}^{\infty}\frac{1}{r(t)}dt=\infty$, we have

$$-\int_{\alpha}^{t}\frac{W'(s)}{W(s)}ds=\int_{\alpha}^{t}\frac{1}{r(s)}ds$$

$$\frac{1}{W(t)}=\frac{1}{W(\alpha)}+\int_{\alpha}^{t}\frac{1}{r(s)}ds$$

$$W(t)=\frac{1}{\frac{1}{W(\alpha)}+k\int_{\alpha}^{t}\frac{1}{r(s)}ds}$$

from where for $t \to \infty$, we obtain

$$\lim_{t\to\infty}W(t)=0\cdot$$

Here , we present some sufficient conditions for (1) to be oscillatory to used the class *X* of functions $H(t,s)$ which comes from Philos, where $H(t,s)>0$, for $t>s\ge t_0$, $H(t,t)=0$, are continuous and have partial derivatives

$$\frac{\partial H(t,s)}{dt} \text{ and } \frac{\partial H(t,s)}{ds}$$

satisfying :

$$\frac{\partial H(t,s)}{\partial s}=-h_2(t,s)\sqrt{H(t,s)} \text{ and } \frac{\partial H(t,s)}{\partial s}=h_1(t,s)\sqrt{H(t,s)} .$$

Theorem 2. Let assumptions A1) – A4) hold and $H\in X$. If there exists $(a,b)\subseteq[t_0,\infty), c\in(a,b)$, such that

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)[q(s)c_1-\frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}}]ds+$$

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)[q(s)c_1+\frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}}]ds>0 \qquad (6)$$

then every solution of eq. (1) is oscillatory.

Proof: Supposed to the contrary, that x *(t)* be a non-oscillatory solution of (1), say $x(t)\neq 0$ on $[t_0,\infty)$.
If we multiply equation (5) by $H^{\alpha+1}(s,t)$ and integrate it from *c* to *t* where $t\in(c,b), s\in(c,t)$ then we have

$$\int_c^t H^{\alpha+1}(t,s)q(s)c_1ds\le-\int_c^t H^{\alpha+1}(t,s)W'(s)ds-\int_c^t H^{\alpha+1}(t,s)\frac{p(s)W(s)}{r(s)}ds-$$
$$-\int_c^t\frac{W^2(s)kH^{\alpha+1}(t,s)}{r(s)}ds$$

$$\int_c^t H^{\alpha+1}(t,s)q(s)c_1ds\le-H^{\alpha+1}(t,s)W(s)/_c^t+\int_c^t W(s)H^{\alpha+1}(t,s)h_1(t,s)\sqrt{H(t,s)}ds-$$

$$-\int_c^t H^{\alpha+1}(t,s)\frac{p(s)W(s)}{r(s)}ds-\int_c^t\frac{W^2(s)kH^{\alpha+1}(t,s)}{r(s)}ds$$

$$\int_c^t H^{\alpha+1}(t,s)q(s)c_1ds\le-H^{\alpha+1}(t,s)W(s)/_c^t-$$
$$-\int_c^t H^{\alpha+1}(t,s)\frac{k}{r(s)}[W^2(s)+(\frac{p(s)}{k}+\frac{(\alpha+1)h_1(t,s)r(s)}{k\sqrt{H(t,s)}}W(s)ds$$

from that

$$\int_c^t H^{\alpha+1}(t,s)q(s)c_1ds\le-H^{\alpha+1}(t,s)W(s)/_c^t+\int_c^t H^{\alpha+1}(t,s)\frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}})^2ds \qquad (7)$$

Let $t\to b\_$ in (7) and dividing it by $H^{\alpha+1}(b,c)$ then what we get is

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)q(s)c_1ds\le W(c)+$$

$$+\frac{1}{H^{\alpha+1}(b,c)}\int_c^t H^{\alpha+1}(t,s)\frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}} \quad (8)$$

By multiplying (5) by $H(s,t)^{\alpha+1}$ and integrate it over (t,c) where $t\in(a,c), s\in(t,c)$ we have

$$\int_t^c H^{\alpha+1}(s,t)q(s)c_1 ds \le -\int_t^c H^{\alpha+1}(s,t)W'(s)ds - \int_t^c H^{\alpha+1}(s,t)\frac{p(s)W(s)}{r(s)}ds -$$

$$-\int_t^c \frac{W^2(s)kH^{\alpha+1}(s,t)}{r(s)}ds$$

from that

$$\int_t^c H^{\alpha+1}(s,t)q(s)c_1 ds \le -H^{\alpha+1}(s,t)W(s)/_t^c +$$

$$+\int_t^c H^{\alpha+1}(s,t)\frac{1}{4kr(s)}(p(s)-(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(s,t)}} \quad (9)$$

Let $t\to b_+$ in (9) and dividing it by $H^{\alpha+1}(b,c)$ we get

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)q(s)c_1 ds \le -W(c)+$$

$$+\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(t,s)\frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}} \quad (10)$$

Adding (8) and (10) we have the following inequality

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)[q(s)c_1 - \frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}}]ds +$$

$$\frac{1}{H^{\alpha+1}(b,c)}\int_c^b H^{\alpha+1}(b,s)[q(s)c_1 + \frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)h_1(s)}{\sqrt{H(t,s)}}]ds \le 0 \quad (11)$$

which contradicts the condition (6) , therefore , every solution of equation (1) is oscillatory .
If for $H(t,s)=(t-s)$, $t\ge s\ge t_0$, we have the following corollary.

Corollary 1.Assuming that condition A1) – A4) hold, then every solution of equation (1) is oscillatory if for $\alpha>0$, the following inequalities hold:

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha+1}}\int_c^t (s-l)^{\alpha+1}[q(s)c_1 - \frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)}{(s-l)}]ds > 0 \quad (12)$$

and

$$\limsup_{t\to-\infty}\frac{1}{t^{\alpha+1}}\int_t^b (l-s)^{\alpha+1}[q(s)c_1 + \frac{1}{4kr(s)}(p(s)+(\alpha+1)\frac{r(s)}{(l-s)}]ds > 0 \quad (13)$$

Proof: For

$H(t,s)=(t-s)$, we have $h_1(t,s)=h_2(t,s)=(t-s)^{-\frac{1}{2}}$ from where similarly to use (6) we get (12) and (13). The proof is complete.

Example 1.Consider the nonlinear differential equation of second order

$$(x'(t))'+(1+\cos^2 t)x'(t)+x(t)(1+x^4(t))(1+(x'(t))^2)=0$$

where $t > 1$.

We can see that $f(x) = x(1 + x^4)$, $f'(x) = 1 + 5x^4 \geq 1$, $g(x') = 1 + x^2 \geq 1$,

for all $x \in \Re$. Equation is oscillatory because the coefficients of equation satisfy the condition given in the theorem 1.

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