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Existence Of Solutions For Nonlinear Fractional Differential Equation With Integral Boundary Conditions

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Abstract. *In this paper we discuss the existence of solutions defined in* $C[0,T]$ *for boundary value problems for a nonlinear fractional differential equation with a integral condition. The results are derived by using the Ascoli-Arzela theorem and Schauder-Tychonoff fixed point theorem.*

***Keywords**: Riemann_Liouville fractional derivative and integral, boundary value problem, nonlinear fractional differential equation, integral condition, Ascoli-Arzela theorem and Schauder-Tychonoff fixed point theorem.*

I. INTRODUCTION

Fractional boundary value problem occur in mechanics and many related fields of engineering and mathematical physics, see Ahmad and Ntouyas [2], Darwish and Ntouyas [4], Hamani, Benchohra and Graef [6], Kilbas, Srivastava and Trujillo [7] and references therein. Various problems has faced in different fields such as population dynamics, blood flow models, chemical engineering and cellular systems that can be modeled to a nonlinear fractional differential equation with integral boundary conditions. Recently, many authors focused on boundary value problems for fractional differential equations, see Ahmad and Nieto [1], Darwish and Ntouyas [4] and the references therein. Some works has been published by many authors on existence and uniqueness of solutions for nonlocal and integral boundary value problems such as Ahmad and Ntouyas [2] and Hamani, Benchohra and Graef [6].

In this paper we prove the existence of the solutions of a nonlinear fractional differential equation with an integral boundary condition at the right end point of $[0,T]$ in $C[0,T]$, where $C[0,T]$ is the space of all continuous functions over $[0,T]$,which results are based on Ascoli-Arzela theorem and Schauder-Tychonoff fixed point theorem.

II. PRELIMINARIES

In this section we introduce definitions, lemmas and theorems which are used throughout this paper. For references see Barrett [3], Kilbas, Srivastava and Trujillo [7] and references therein.

Definition 2.1. Let f be a function which is defined almost everywhere on $[a,b]$. For $\alpha > 0$, we define:

$$ {}_a^bD^{-\alpha} f = \frac{1}{\Gamma(\alpha)}\int_a^b f(t)(b-t)^{\alpha-1}dt $$

provided that this integral exists in Lebesgue sense, where Γ is the gamma function.

Lemma 2.2. Assume that $f \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$, then

$$ D_{0^+}^{-\alpha} D_{0^+}^{\alpha} f(t) = f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n} $$

for some $C_i \in R$; i =1, 2, …, n, where n is the smallest integer greater than or equal to α .

Lemma 2.3. Let $\alpha, \beta \in R, \beta > -1$. If $x > a$, then

$$ {}_a^xI^{\alpha} \frac{(x-a)^{\beta}}{\Gamma(\beta+1)} = \begin{cases} \dfrac{(x-a)^{\beta+\alpha}}{\Gamma(\alpha+\beta+1)} & ;\alpha+\beta \neq negative\ \ integer \\ 0 & ;\alpha+\beta = negative\ \ integer \end{cases} $$

Lemma 2.4. The following relation ${}_a^xD^{-\alpha}\, {}_a^xD^{-\beta} f = {}_a^xD^{-(\alpha+\beta)} f$ holds if

a. $\alpha > 0$, $\beta > 0$ and the function $f(x) \in C$ on a closed interval $[a,b]$.

b. $\alpha \le 0$ or $\alpha + n > 0$, $\beta > 0$ and the function $f(x) \in C^{(n)}$ on a closed interval $[a,b]$.

Lemma 2.5. If $\alpha > 0$ and $f(x)$ is continuous on $[a,b]$, then ${}_a^xD^{-\alpha} f(x)$ exists and it is continuous with respect to *x* on $[a,b]$.

Theorem 2.6. (The Arzela Ascoli Theorem)
Let F be an equicontinuous, uniformly bounded family of real valued functions f on an interval I (finite or infinite).Then F contains a uniformly convergent sequence of function f_n, converging to a function $f \in C(I)$ where $C(I)$ denotes the space of all continuous bounded functions on I . Thus any sequence in F contains a uniformly bounded convergent subsequence on I and consequently F has a compact closure in $C(I)$.

Theorem 2.7. (Shauder-Tychnoff Fixed Point Theorem)
Let B be a locally convex, topological vector space. Let Y be a compact, convex subset of B and T a continuous map of Y into itself. Then T has a fixed point $y \in Y$.

III. MAIN RESULT

The statements and proofs for the main results are carried out in this section.

Lemma 3.1. Let $f(t, x(t))$ and $h(t, x(t))$ belong to $C[0,T]$, and $1 < \alpha \le 2$, then the solution of

$$D^{\alpha} x(t) = f\left(t, x(t), \int_0^T h(t,\tau,x(\tau))d\tau\right), \quad t \in (0,T) \tag{3.1}$$

$$x^{(\alpha-2)}(0) = 0 \tag{3.2}$$

$$x^{(\alpha-1)}(T) = a\int_0^{\eta} x(\tau)\,d\tau \tag{3.3}$$

Where $\eta \in (0,T)$ and *a* is a constant, is given by

$$x(t) = \frac{t^{\alpha-1}}{\theta - \Gamma(\alpha)} \int_0^T f(t, x(t), \int_0^T h(t,\tau,x(\tau))d\tau)\,dt - \frac{a t^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))} \int_0^{\eta} (\eta-\tau)^{\alpha} f(\tau, x(\tau),$$
$$\int_0^T h(\tau,s,x(s))ds)\,d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\left(\tau, x(\tau), \int_0^T h(\tau,s,x(s))ds\right)d\tau$$

Where $\theta = \frac{a}{\alpha}\eta^{\alpha}$ and $\theta \neq \Gamma(\alpha)$.

Proof. Operate both sides of equation (3.1) by the operator $D^{-\alpha}$, to obtain

$$D^{-\alpha} D^{\alpha} x(t) = D^{-\alpha} f\left(t, x(t), \int_0^T h(t,\tau,x(\tau))d\tau\right)$$

From Lemma (2.1), we get

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + D^{-\alpha} f\left(t, x(t), \int_0^T h(t,\tau,x(\tau))d\tau\right) \tag{3.4}$$

Now, operate both sides of equation (3.4) by the operator $D^{\alpha-1}$, to have

$$D^{\alpha-1} x(t) = D^{\alpha-1} C_1 t^{\alpha-1} + D^{\alpha-1} C_2 t^{\alpha-2} + D^{\alpha-1} D^{-\alpha} f\left(t, x(t), \int_0^T h(t,\tau,x(\tau))d\tau\right) \tag{3.5}$$

From Lemma (2.2) and Lemma (2.3), $D^{\alpha-1} C_1 t^{\alpha-1} = C_1 \Gamma(\alpha)$, $D^{\alpha-1} C_2 t^{\alpha-2} = 0$ and

$$D^{\alpha-1}D^{-\alpha} f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right)=D^{-1}\ f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right)$$

Therefore equation (3.5) can be written as

$$D^{\alpha-1}x(t)=C_1\Gamma(\alpha)+{}_0^tD^{-1}f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right) \tag{3.6}$$

Now, operating on both sides of equation (3.4) by the operator $D^{\alpha-2}$ and using again Lemma (2.2) and Lemma (2.3), equation (3.4) takes the form

$$D^{\alpha-2}x(t)=C_1\Gamma(\alpha)t+C_2\Gamma(\alpha-1)+{}_0^tD^{-2}f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right) \tag{3.7}$$

Now from the condition (3.2) and equation (3.7), it follows that $C_2=0$, and from the condition (3.3) and equation (3.6) we get

$$a\int_0^\eta x(\tau)d\tau=C_1\Gamma(\alpha)+\int_0^T f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right)dt$$

$$aC_1\int_0^\eta \tau^{\alpha-1}d\tau+\frac{a}{\Gamma(\alpha)}\int_0^\eta\int_0^\tau(\tau-s)^{\alpha-1}f\left(s,x(s),\int_0^T h(s,z,x(z))dz\right)dsd\tau=C_1\Gamma(\alpha)+\int_0^T f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right)dt$$

$$C_1=\frac{1}{\theta-\Gamma(\alpha)}\left[\int_0^T f\left(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau\right)dt-\frac{a}{\Gamma(\alpha+1)}\int_0^\eta(\eta-\tau)^\alpha f\left(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds\right)d\tau\right]$$

Where $\theta=\frac{a}{\alpha}\eta^\alpha$ and $\theta\neq\Gamma(\alpha)$, therefore the solution of the given boundary value problem takes the form

$$x(t)=\frac{t^{\alpha-1}}{\theta-\Gamma(\alpha)}\int_0^T f(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau)\,dt-\frac{at^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}\int_0^\eta(\eta-\tau)^\alpha f(\tau,x(\tau),$$
$$\int_0^T h(\tau,s,x(s))ds)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\left(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds\right)d\tau.$$

Theorem: Assume that $f(t,x(t))$ and $h(t,x(t))$ belong to $C[0,T]$, then the fractional boundary value problem (3.1-3) has a unique solution on $[0,T]$.

Proof: Let $X=\{x(t);x(t)\in C[0,T]\}$ and the mapping $T:C[0,T]\to C[0,T]$ defined by

$$Tx(t)=\frac{t^{\alpha-1}}{\theta-\Gamma(\alpha)}\left[\int_0^T f(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau)dt-\frac{a}{\Gamma(\alpha+1)}\int_0^\eta(\eta-\tau)^\alpha\right.$$
$$\left.f(\tau,x(\tau),\int_0^T h(\tau,s,\ x(s))ds)d\tau\right]+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau \tag{3.8}$$

in order to apply the Schauder-Tychonoff fixed point theorem, we should prove the following steps

Step1: T maps X into itself.

Let $x\in X$, since f is continuous on $[0,T]$, it guarantees that all the terms on (3.8) are continuous. Thus T maps Y into itself

Step2: T is a continuous mapping on X.

Let $\{x_n(t)\}_{n=1}^\infty$ be a sequence in X such that $\lim_{n\to\infty}x_n(t)=x(t)$ where $x(t)\in C[0,T]$, consider

$$\|Tx_n(t)-Tx(t)\|=$$
$$\left\|\frac{t^{\alpha-1}}{\theta-\Gamma(\alpha)}\left\{\int_0^T(f(t,x_n(t),\int_0^T h(t,\tau,x_n(\tau))d\tau-f(t,x(t),\int_0^T h(t,\tau,x(\tau))d\tau)dt\right.\right.$$

$$-\frac{a}{\Gamma(\alpha+1)}\int_0^{\eta}(\eta-\tau)^{\alpha}[f(\tau,x_n(\tau),\int_0^T h(\tau,s,x_n(s))ds)-f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau]\Bigg\}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-\tau)^{\alpha-1}[f(\tau,x_n(\tau),\int_0^T h(\tau,s,x_n(s))ds)-f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)]d\tau\Bigg\| \qquad (3.9)$$

the right hand side of the equation (3.9) tends to zero as n tends to infinity, since f is a continuous function and the sequence $\{x_n(t)\}_{n=1}^{\infty}$ converges to $x(t)$, that is

$$|x_n(t)-x(t)|\to 0 \qquad \text{as } n\to\infty$$

Also since f is bounded hence by Lebesgue's dominated convergence theorem we have

$$\|Tx_n(t)-Tx(t)\|\to 0 \quad \text{as } n\to\infty$$

therefore T is a continuous mapping on X .

Step3: The closure of $TX=\{Tx(t)\ ;\ x(t)\in X\}$ is compact.

To prove step3 we will prove that the family TX is uniformly bounded and equicontinuous. TX is uniformly bounded as shown in step1, for proving the equicontinuity, let $t_1,t_2\in(0,T]$ such that $t_1<t_2$, then

$$\begin{aligned}|Tx(t_2)-Tx(t_1)|=\Bigg|&\frac{t_2^{\alpha-1}}{\theta-\Gamma(\alpha)}\int_0^T f(t_2,x(t_2),\int_0^T h(t_2,\tau,x(\tau))d\tau)dt_2\\&-\frac{a t_2^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}\int_0^{\eta}(\eta-\tau)^{\alpha} f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau\\&+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau\\&-\frac{t_1^{\alpha-1}}{\theta-\Gamma(\alpha)}\int_0^T f(t_1,x(t_1),\int_0^T h(t_1,\tau,x(\tau))d\tau)dt_1\\&+\frac{a t_1^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}\int_0^{\eta}(\eta-\tau)^{\alpha} f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau\\&-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_1-\tau)^{\alpha-1}f(\tau,x(\tau),\int_0^T h(\tau,s,x(s))ds)d\tau\Bigg|\end{aligned}$$

By continuity of f on $[0,T]$ there exists a positive constant M such that

$$\begin{aligned}|Tx(t_2)-Tx(t_1)|\le\Bigg|&\frac{t_2^{\alpha-1}}{\theta-\Gamma(\alpha)}\int_0^T Mdt_2-\frac{a t_2^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}\int_0^{\eta}(\eta-\tau)^{\alpha}Md\tau\\&+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-\tau)^{\alpha-1}Md\tau-\frac{t_1^{\alpha-1}}{\theta-\Gamma(\alpha)}\int_0^T Mdt_1\\&+\frac{a t_1^{\alpha-1}}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}\int_0^{\eta}(\eta-\tau)^{\alpha}Md\tau-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}Md\tau\Bigg|\\=\Bigg|&\frac{MT}{\theta-\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{aM}{\Gamma(\alpha+1)(\theta-\Gamma(\alpha))}(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^{\eta}(\eta-\tau)^{\alpha}d\tau\\&+\frac{M}{\Gamma(\alpha)}[\int_0^{t_2}(t_2-\tau)^{\alpha-1}d\tau-\int_0^{t_1}(t_1-\tau)^{\alpha-1}d\tau]\Bigg|\\=\Bigg|&\frac{MT}{\theta-\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{aM\eta^{\alpha+1}}{\Gamma(\alpha+2)(\theta-\Gamma(\alpha))}(t_2^{\alpha-1}-t_1^{\alpha-1})\end{aligned}$$

$$+\frac{M}{\Gamma(\alpha)}[\int_0^{t_1}[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]d\tau+\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau]\Bigg|$$

$$=\Bigg|\frac{MT}{\theta-\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{aM\eta^{\alpha+1}}{\Gamma(\alpha+2)(\theta-\Gamma(\alpha))}(t_2^{\alpha-1}-t_1^{\alpha-1})$$

$$+\frac{M}{\Gamma(\alpha)}\left[\frac{t_2^{\alpha}}{\alpha}-\frac{(t_2-t_1)^{\alpha}}{\alpha}+\frac{(t_2-t_1)^{\alpha}}{\alpha}-\frac{t_1^{\alpha}}{\alpha}\right]\Bigg|$$

$$|Tx(t_2)-Tx(t_1)|\leq$$

$$\left|\frac{MT}{\theta-\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{aM\eta^{\alpha+1}}{\Gamma(\alpha+2)(\theta-\Gamma(\alpha))}(t_2^{\alpha-1}-t_1^{\alpha-1})+\frac{M}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})\right|$$

when t_1 tends to t_2, with $|t_1-t_2|<\delta$, we have $|Tx(t_2)-T\,x(t_1)|<\varepsilon$, which proves that the family TX is equicontinuous. Thus by Ascoli-Arzela theorem, TX has a compact closure. In view of step1, step2 and step3, the Schauder-Tychonoff fixed point theorem guarantees that T has at least one fixed point $x\in X$, that is $Tx(t)=x(t)$.

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