Research Inventy: International Journal of Engineering And Science
Vol.9, Issue 1 (January. 2019), PP 60-65
Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com

Fixed points under for Pseudo Compatible Maps

Dr. Maheshwari P.G.
Department of Mathematics
Government First Grade College, Vijayanagar, Bangalore

Abstract— *In this paper, we have proved some common fixed point theorems by using the new notion of f-reciprocal continuity and pseudo compatibility, a generalized form of occasional weak compatibility, under contractive condition as well as Lipschitz type condition that extend the scope of the study of common fixed point theorems from the class of compatible mappings to a wider class of mappings that includes non compatible and discontinuous mappings. The suitable examples are demonstrated to exhibit the utility of the main results. These results extend and generalize the many results in the literature.*

Keywords—*Pseudo compatible maps, f-reciprocal continuity, Fixed point, compatible maps, non-compatible maps.*
Mathematics Subject Classification: *54H25, 47H10*

I. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays a significant role in non-linear analysis as many real-world problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of non-linear maps. Common fixed point theorem requires commutativity, continuity, completeness together with a suitable condition on containment of ranges of involved maps beside an appropriate contraction condition. Thus, research in this field is aimed at weakening one or more of these conditions. In this regard, the problem "whether there exists a contractive definition which is strong enough to generate a fixed point, but which does not force the map to be continuous" was reiterated by Rhoades [1] and has remained open for more than a decade. This problem was settled in the affirmative by Pant [2], by introducing the notion of reciprocal continuity, which is mainly applicable to the setting of compatible mappings.

Definition 1.1:[10] Two self maps f and g of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(\mathrm{fgx_n}, \mathrm{gfx_n}) = 0$, whenever $\{\mathrm{x_n}\}$ is a sequence in X such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X.
Thus the mappings f and g will be non compatible if there exists at least one sequence $\{\mathrm{x_n}\}$ such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X but $\lim_{n\to\infty} d(\mathrm{fgx_n}, \mathrm{gfx_n})$ is either nonzero or nonexistent.
Definition 1.2: Two self mappings f and g of a metric space (X, d) are called reciprocally continuous if $\lim_{n\to\infty} \mathrm{fgx_n} = \mathrm{ft}$ and $\lim_{n\to\infty} \mathrm{gfx_n} = \mathrm{gt}$ whenever $\{\mathrm{x_n}\}$ is a sequence in X such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X.
If f and g are both continuous then they are obviously reciprocally continuous but the converse is not true.
Later, Pant [3] succeeded in generalizing the concept of reciprocal continuity to weak reciprocal continuity and they were able to demonstrate that above mentioned problem has affirmative answer for non compatible mappings also. More importantly, in the case of non compatible mappings the problem has an affirmative answer not only under contractive conditions but also under non expansive or Lipschitz type conditions.

Definition 1.3: Two self mappings *f* and *g* of a metric space *(X, d)* are called weakly reciprocally continuous if $\lim_{n\to\infty} \mathrm{fgx_n} = \mathrm{ft}$ or $\lim_{n\to\infty} \mathrm{gfx_n} = \mathrm{gt}$ whenever $\{\mathrm{x_n}\}$ is a sequence in X such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X.
If f and g are reciprocally continuous then they are obviously weakly reciprocally continuous, but the converse is not true. Many interesting works on weak reciprocal continuity have come through by many authors (see [6]-[8]).

As a development, Pant et.al.[4] have introduced two more generalized concepts, *g*-reciprocal continuity which is a generalization of continuity, but independent of reciprocal continuity (see examples in [5]) and Pseudo compatible mappings, a proper generalization of occasionally weakly compatible.

Definition 1.4: Two self mappings f and g of a metric space (X, d) are called g- reciprocally continuous if and only if $\lim_{n\to\infty} \mathrm{ffx_n} = \mathrm{ft}$ and $\lim_{n\to\infty} \mathrm{gfx_n} = \mathrm{gt}$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X.

Definition 1.5: Let f and g be self mappings of a metric space (X,d). Then for a sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} \mathrm{fy_n} = \lim_{n\to\infty} \mathrm{gy_n}$, a sequence $\{z_n\}$ will be called an associated sequence if $\mathrm{fy_n} = \mathrm{gz_n}$ or $\mathrm{gy_n} = \mathrm{fz_n}$ and $\lim_{n\to\infty} \mathrm{fz_n} = \lim_{n\to\infty} \mathrm{gz_n}$.

Definition 1.6: Two self mappings f and g of a metric space (X,d) will be defined to be pseudo compatible if and only if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n}$ is nonempty, there exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} \mathrm{fy_n} = \lim_{n\to\infty} \mathrm{gy_n} = \mathrm{t}$ (say), $\lim_{n\to\infty} \mathrm{d(fgy_n, gfy_n)} = 0$ and $\lim_{n\to\infty} \mathrm{d(fgz_n, gfz_n)} = 0$ for any associated sequence $\{z_n\}$ of $\{y_n\}$.

By using these concepts Pant in [4] obtained the following results.

Theorem 1.7: Let f and g be g-reciprocally continuous self mappings of a complete metric space (X,d) such that

(i) $\mathrm{fX} \subseteq \mathrm{gX}$

(ii) $\mathrm{d(fx, fy)} \leq \mathrm{k\, d(gx, gy)}$, $\mathrm{k} \in [0,1)$.

If f and g are pseudo compatible, then f and g have a unique common fixed point.

Theorem 1.8: Let f and g be g-reciprocally continuous non compatible self mappings of a metric space (X,d) such that

(i) $\mathrm{fX} \subseteq \mathrm{gX}$

(ii) $\mathrm{d(fx, fy)} < max\left\{d(\mathrm{gx, gy}), \frac{\mathrm{k[d(fx,gx)+d(fy,gy)]}}{2}, \frac{\mathrm{[d(fx,gy)+d(fy,gx)]}}{2}\right\}$, $\quad 1 \leq \mathrm{k} < 2$.

(iii) $\mathrm{d(x, fx)} \neq \max(\mathrm{d(x, gx), d(fx, gx)})$,

whenever right-hand side is nonzero. If f and g are pseudo compatible, then f and g have a unique common fixed point.

Motivated by the work of Pant in [4], Giniswamy et al. [9] have defined f-reciprocal continuity, a generalization of continuity but independent of both reciprocal and g-reciprocal continuity (see examples in [9]).

Definition 1.9: Two self mappings f and g of a metric space (X, d) are called f-reciprocally continuous if and only if $\lim_{n\to\infty} \mathrm{fgx_n} = \mathrm{ft}$ and $\lim_{n\to\infty} \mathrm{ggx_n} = \mathrm{gt}$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathrm{fx_n} = \lim_{n\to\infty} \mathrm{gx_n} = \mathrm{t}$ for some t in X.

The following are the main results proved in [9].

Theorem 1.10: Let f and g be f-reciprocally continuous self mappings of a metric space (X,d) such that

(i) $\mathrm{fX} \subseteq \mathrm{gX}$ and fX is complete

(ii) $\mathrm{d(fx, fy)} \leq \mathrm{a\, d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy)}$

with $\mathrm{a, b, c} \in [0, 1)$ and a +b+c <1.

If f and g are pseudo compatible, then f and g have a unique common fixed point.

Theorem 1.11: Let f and g be f-reciprocally continuous non compatible self mappings of a metric space (X, d) such that

(i) $\mathrm{fX} \subseteq \mathrm{gX}$

(ii) $\mathrm{d(fx, fy)} < \max\left\{\mathrm{kd(gx, gy)}, \frac{\mathrm{k}}{2}[\mathrm{d(fx, gx) + d(fy, gy)}], \frac{\mathrm{k}}{2}[\mathrm{d(fx, gy) + d(fy, gx)}]\right\}$,

$\forall \mathrm{x} \neq \mathrm{y}$ where $0 < k < 1$.

If f and g are pseudo compatible, then f and g have a unique common fixed point.

In this paper, we have proved some common fixed point theorems by using this f-reciprocal continuity and pseudo compatibility under contractive condition as well as Lipschitz type condition that extend the scope of the study of common fixed point theorems from the class of compatible mappings to a wider class of mappings that includes non compatible and discontinuous mappings. The suitable examples are demonstrated to exhibit the utility of the main results. These results extend and generalize the results of Pant [4], Giniswamy et al. [9] and many more results in the literature.

II. MAIN RESULT

We now state and prove our first main result.

Theorem 2.1: *Let f and g be f-reciprocally continuous self mappings of a metric space (X,d) such that*

(i) $\overline{fX} \subseteq gX$

(ii) $d(fx, fy) \leq a\, d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy) + e\, [d(fx, gy) + d(fy, gx)]$

with $a, b, c, e \in [0, 1)$ *and a +b+c+2e <1.*

If f and g are pseudo compatible, then f and g have a unique common fixed point.

Proof:

Let x_0 be any point in X. Since $\overline{fX} \subseteq gX$, there exists a sequence of points $x_0, x_1, x_2, \ldots, x_n, \ldots$ such that x_{n+1} is in the preimage under g of fx_n.

$$\text{i.e. } fx_0 = gx_1,\ fx_1 = gx_2, \ldots, fx_n = gx_{n+1}, \ldots$$

Define a sequence $\{S_n\}$ in X by

$$S_n = fx_n = gx_{n+1} \quad \text{for } n = 0,1,2,\ldots$$

Clearly $\{S_n\}$ is a sequence in $\overline{fX}$.

Now, we claim that $\{S_n\}$ is a Cauchy sequence in $\overline{fX}$. Consider

$$\begin{aligned} d(S_n, S_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\leq a\,d(gx_n, gx_{n+1}) + b\,d(fx_n, gx_n) + c\,d(fx_{n+1}, gx_{n+1}) + e[d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)] \\ &= a\,d(S_{n-1}, S_n) + b\,d(S_n, S_{n-1}) + c\,d(S_{n+1}, S_n) + e[d(S_n, S_n) + d(S_{n+1}, S_{n-1})] \end{aligned}$$

$$\text{i.e. } d(S_n, S_{n+1}) \leq k\,d(S_{n-1}, S_n) \leq k^n\,d(S_0, S_1), \qquad \text{where } k = \left(\frac{a+b+e}{1-c-e}\right) < 1.$$

Also for every integer $p > 0$, we have

$$\begin{aligned} d(S_n, S_{n+p}) &\leq d(S_n, S_{n+1}) + d(S_{n+1}, S_{n+2}) + \ldots + d(S_{n+p-1}, S_{n+p}) \\ &\leq k^n(1 + k + k^2 + \ldots + k^{p-1})\,d(S_0, S_1) \\ &\leq \left(\frac{1}{1-k}\right) k^n\,d(S_0, S_1) \end{aligned}$$

That is $d(S_n, S_{n+p}) \to 0$ as $n \to \infty$. Therefore $\{S_n\}$ is a Cauchy sequence in $\overline{fX}$.

Hence, there exists a point $t \in \overline{fX}$ such that $S_n \to t$ as $n \to \infty$.

Moreover, $S_n = fx_n = gx_{n+1} \to t$.

Now f and g are pseudo compatible implies there exists a sequence $\{y_n\}$ such that $fy_n \to u$, $gy_n \to u$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.

Since $\overline{fX} \subseteq gX$, for each y_n there exists a z_n in X such that $fy_n = gz_n$ $\forall n$.

Now we prove that $fz_n \to u$. Consider

$$d(fy_n, fz_n) \leq a\,d(gy_n, gz_n) + b\,d(fy_n, gy_n) + c\,d(fz_n, gz_n) + e[d(fy_n, gz_n) + d(fz_n, gy_n)]$$

On letting $n \to \infty$ we get

$$d(u, fz_n) \leq a\,d(u,u) + b\,d(u,u) + c\,d(fz_n, u) + e[d(u,u) + d(fz_n, u)]$$

$$\text{i.e. } (1-c-e)d(u, fz_n) \leq 0, \quad \text{which gives } fz_n \to u, \text{ since } c + e < 1.$$

Therefore $\{y_n\}$ and $\{z_n\}$ are associated sequences and $\lim_{n\to\infty} d(fgz_n, gfz_n) = 0$.

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n = u.$$

Further, f- reciprocal continuity of f and g implies that $fgy_n \to fu$ and $ggy_n \to gu$.

Since $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$, we have $gfy_n = ggz_n \to fu$. Similarly, $fgz_n \to fu$ and $ggz_n \to gu$. Hence $fu = gu$.

Now we prove that $fu = u$. Consider

$$d(fu, fz_n) \leq a\,d(gu, gz_n) + b\,d(fu, gu) + c\,d(fz_n, gz_n) + e[d(fu, gz_n) + d(fz_n, gu)]$$

On letting $n \to \infty$ we get

$$d(fu, u) \leq a\,d(fu, u) + b\,d(fu, fu) + c\,d(u,u) + e[d(fu, u) + d(u, fu)]$$

$$\text{i.e. } (1-a-2e)d(fu, u) \leq 0, \quad \text{which gives } u = fu = gu \text{ since } a + 2e < 1.$$

Therefore u is a common fixed point of f and g.

To prove the uniqueness, let u and v be two common fixed points of f and g. Then $u = fu = gu$ and $v = fv = gv$. Consider

$$d(u,v) = d(fu, fv) \leq a\,d(gu, gv) + b\,d(fu, gu) + c\,d(fv, gv) + e[d(fu, gv) + d(fv, gu)]$$

on letting $n \to \infty$ we get $d(u,v) \leq (a+2e)d(u,v)$, which gives $u = v$ since $a + 2e < 1$.

Therefore u is the unique common fixed point of f and g.

The above theorem is illustrated by the following example.

Example: Let $X = [0,10]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = \frac{x+3}{2} \text{ if } x \leq 3, \qquad fx = 1 \text{ if } x > 3$$

$$gx = \frac{2x+3}{3} \text{ if } x \leq 3, \qquad gx = 8 \text{ if } x > 3$$

Then f and g satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at $x = 3$. Further, f and g satisfy the contraction condition (ii) for

$$a = \frac{1}{4}, b = \frac{1}{3}, c = \frac{1}{6} \text{ and } e = \frac{1}{12}.$$

The mappings f and g are f-reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$ and $gx_n \to t$ for some t.

Then $t = 3$ and either $x_n = 3$ for each n or $x_n = 3 - \frac{1}{n}$.

If $x_n = 3$ for each n then $fx_n = 3$, $gx_n = 3$, $fgx_n = f3 = 3$ and $ggx_n = g3 = 3$.

If $x_n = 3 - \frac{1}{n}$ then $fx_n = 3$, $gx_n = 3 - \frac{2}{3n} \to 3$, $fgx_n = f\left(3 - \frac{2}{3n}\right) = 3 - \frac{1}{3n} \to 3 = f3$ and $ggx_n = g\left(3 - \frac{2}{3n}\right) = 3 - \frac{4}{9n} \to 3 = g3$.

$$\text{Thus } \lim_{n\to\infty} fgx_n = f3 \text{ and } \lim_{n\to\infty} ggx_n = g3.$$

Hence f and g are f-reciprocally continuous mappings.

Also f and g are pseudo compatible. To see this consider the sequence $\{x_n\} = \left\{3 - \frac{1}{n}\right\}$. Then $fx_n \to 3$ and $gx_n \to 3$. Consider another sequence $\{y_n\} = 3$ for all n. Then $fy_n \to 3$, $gy_n \to 3$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.

If $\{z_n\}$ is an associated sequence of $\{y_n\}$ such that $fy_n = gz_n$ $\forall n$ and $\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n$, then $z_n = 3$ $\forall n$ and $\lim_{n\to\infty} d(fgz_n, gfz_n) = 0$.

Corollary 2.2: *Let f and g be f-reciprocally continuous self mappings of a metric space (X,d) such that*

(i) $\overline{fX} \subseteq gX$

(ii) $d(fx, fy) \leq a\, d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy)$

with $a, b, c \in [0, 1)$ *and* $a + b + c < 1$.

If f and g are pseudo compatible, then f and g have a unique common fixed point.

Remark 2.3: Note that the Theorem 2.1 generalizes Theorem 1.7 and Theorem 1.10 and Corollary 2.2 is a sharpened version of Theorem 1.10.

It is well known that strict contractive conditions do not ensure the existence of fixed points unless very strong conditions like compactness are assumed. But the next result demonstrates that the generalized strict contractive condition ensures the existence of common fixed point under the notion of *f*-reciprocal continuity.

Theorem 2.4: *Let f and g be f-reciprocally continuous non compatible self mappings of a metric space (X, d) such that*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) < k \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$

$\forall x \neq y$ *where* $0 < k < 1$.

If f and g are pseudo compatible, then f and g have a unique common fixed point.

Proof:

Since f and g are non compatible mappings, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but either $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$ or the limit does not exist.

Now f and g are pseudo compatible implies there exists a sequence $\{y_n\}$ such that $fy_n \to u$, $gy_n \to u$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.

Since $fX \subseteq gX$, for each y_n there exists a, z_n in X such that $fy_n = gz_n$ $\forall n$.

Now we prove that $fz_n \to u$. Consider,

$$d(fy_n, fz_n) < k \max\{d(gy_n, gz_n), d(fy_n, gy_n), d(fz_n, gz_n), d(fy_n, gz_n), d(fz_n, gy_n)\}$$

on letting $n \to \infty$ we get

$$d(u, fz_n) \leq k \max\{\{d(u,u), d(u,u), d(fz_n, u), d(u,u), d(fz_n, u)\}\}$$

i.e. $(1 - k)d(u, fz_n) \leq 0$, which gives $fz_n \to u$ since $k < 1$.

Therefore $\{y_n\}$ and $\{z_n\}$ are associated sequences and $\lim_{n\to\infty} d(fgz_n, gfz_n) = 0$. Then

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n = u.$$

Further, f- reciprocal continuity of f and g implies that $fgy_n \to fu$ and $ggy_n \to gu$.

Since $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$, we have $gfy_n = ggz_n \to fu$. Similarly, $fgz_n \to fu$ and $ggz_n \to gu$. Hence $fu = gu$. Now we prove that $fu = u$. Consider

$$d(fu, fz_n) < k \max\{d(gu, gz_n), d(fu, gu), d(fz_n, gz_n), d(fu, gz_n), d(fz_n, gu)\}$$

on letting $n \to \infty$ we get

$$d(fu, u) < k \max\{d(fu, u), d(fu, fu), d(u, u), d(fu, u), d(u, fu)\}$$

Thus $(1-k)d(fu, u) \leq 0$, which gives $u = fu = gu$ since $k < 1$.
Therefore u is a common fixed point of f and g.
To prove the uniqueness, let u and v be two common fixed points of f and g. Then $u = fu = gu$ and $v = fv = gv$.

Now we prove that $u = v$: Suppose that $u \neq v$, then

$$d(u, v) = d(fu, fv) < k \max\{d(gu, gv), d(fu, gu), d(fv, gv), d(fu, gv), d(fv, gu)\}$$

on letting $n \to \infty$ we get $d(u, v) < kd(u, v) < d(u, v)$, a contradiction. Therefore $u = v$.
Hence u is the unique common fixed point of f and g.

Now we present an example to illustrate Theorem 2.4.

Example: Let $X = [0, 10]$ and d be the usual metric on X. Define $f, g: X \to X$ by

$$fx = 4 - \frac{x}{3} \text{ if } x \leq 3, \ fx = 1 \text{ if } x > 3$$

$$gx = \frac{4x+3}{5} \text{ if } x \leq 3, \ gx = x - \frac{3}{x} \text{ if } x > 3$$

Then f and g satisfy all the conditions of Theorem 2.4 and have a unique common fixed point at $x = 3$. Further, f and g satisfy the contraction condition (ii) for $k = \frac{1}{2}$.
The mappings f and g are f-reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$ and $gx_n \to t$ for some t.
Then $t = 3$ and either $x_n = 3$ for each n or $x_n = 3 - \frac{1}{n}$.
If $x_n = 3$ for each n, then $fx_n = 3$, $gx_n = 3$, $fgx_n = f3 = 3$ and $ggx_n = g3 = 3$.
If $x_n = 3 - \frac{1}{n}$ then $fx_n \to 3$, $gx_n \to 3$, $fgx_n = f\left(3 - \frac{4}{5n}\right) \to 3 = f3$,
$ggx_n = g\left(3 - \frac{4}{5n}\right) \to 3 = g3$ and $gfx_n = g\left(3 + \frac{1}{3n}\right) \to 2 \neq g3$.
Thus $\lim_{n\to\infty} fgx_n = f3$ and $\lim_{n\to\infty} ggx_n = g3$ but $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$.
Hence f and g are f-reciprocally continuous and non compatible mappings.
Also f and g are pseudo compatible. To see this, consider the sequence $\{x_n\} = \left\{3 - \frac{1}{n}\right\}$. Then $fx_n \to 3$ and $gx_n \to 3$. Consider another sequence $\{y_n\} = 3 \ \forall n$. Then $fy_n \to 3$, $gy_n \to 3$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.
If $\{z_n\}$ is an associated sequence of $\{y_n\}$ such that $fy_n = gz_n \ \forall n$ and $\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n$, then $z_n = 3 \ \forall n$ and $\lim_{n\to\infty} d(fgz_n, gfz_n) = 0$.

Corollary 2.5: *Let f and g be f-reciprocally continuous non compatible self mappings of a metric space (X, d) such that* (i) $fX \subseteq gX$
(ii) $d(fx, fy) < k\, max\{d(gx, gy), d(fx, gx), +d(fy, gy), d(fx, gy) + d(fy, gx)\}$
$\forall x \neq y$ *where* $0 < k < 1$.
If f and g are pseudo compatible, then f and g have a unique common fixed point.

Remark 2.6: Note that the Theorem 2.4 generalizes Theorem 1.8 and Theorem 1.11 and Corollary 2.5 is a sharpened version of Theorem 1.11.

Remark 2.7: The results established in this paper ensure the existence of common fixed points without assuming the continuity condition. Thus we provide more answers to the open problem posed by Rhoades [1].

REFERENCES

[1] Rhoades B.E., 1982, Contractive definitions and continuity, Contemporary Math., 72, 233-245.
[2] Pant R.P., 1998, Common fixed points for four mappings, Bull.Cal.Math.Soc., 90, 281-286.
[3] Pant R.P.,Bisht R.K.,Arora D., 2011, Weak reciprocal continuity and fixed point theorems, Ann. Univ.Ferrara, 57(1), 181-190.
[4] Pant R.P.,Bisht R.K., 2014, Common fixed points of pseudo compatible mappings, RACSAM, 10, doi:10.1007/S13398-013-0119-5.
[5] Joshi R.U.,Bisht R.K.,Bhatt A., 2012, Common fixed points under strict contractive condition and g-reciprocal continuity , J.Int.Acad.Phys.Sci., 16(1), 1-6.
[6] Bisht R.K.,Joshi R.U., 2012, Common fixed point theorems for weakly reciprocally continuous maps, J.Ind.Math.Soc., 79,(1-4), 1-12.

[7] Manro S.,Kumam P., 2012, Common fixed point theorems for expansion mappings in various abstract spaces using the concept of weak reciprocal continuity, Fixed point theory and Applications, 2012:221.
[8] Giniswamy, Jeyanthi C.,Maheshwari P.G., 2012, Fixed point theorems for compatible and noncompatible self maps, J.Int.Acad.Phys.Sci., 16(1), 13-26.
[9] Giniswamy, Maheshwari P.G., 2014, Fixed point theorems on f-reciprocal continuity, Aryabhatta J.Math.Inf., 6(1),109-114.
[10] Jungck G., 1986, Compatible mappings and common fixed points, Int.J.Math.Math.Sci., 9, 771-779.