Higher Separation Axioms via Semi*-open sets

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Abstract: The purpose of this paper is to introduce new separation axioms semi*-regular, semi*-normal, s*-regular, s**-normal using semi*-open sets and investigate their properties. We also study the relationships among themselves and with known axioms regular, normal, semi-regular and semi-normal.

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I. INTRODUCTION

Separation axioms are useful in classifying topological spaces. Maheswari and Prasad [8, 9] introduced the notion of s-regular and s-normal spaces using semi-open sets. Dorsett [3, 4] introduced the concept of semi-regular and semi-normal spaces and investigate their properties.

In this paper, we define semi*-regular, semi*-normal, s*-regular and s**-normal spaces using semi*-open sets and investigate their basic properties. We further study the relationships among themselves and with known axioms regular, normal, semi-regular and semi-normal.

II. PRELIMINARIES

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), Cl(A) and Int(A) respectively denote the closure and the interior of A in X.

Definition 2.1[7]: A subset A of a topological space (X, τ) is called
(i) generalized closed (briefly g-closed) if Cl(A) ⊆ U whenever A ⊆ U and U is open in X.
(ii) generalized open (briefly g-open) if X \ A is g-closed in X.

Definition 2.2: Let A be a subset of X. Then
(i) generalized closure[5] of A is defined as the intersection of all g-closed sets containing A and is denoted by Cl*(A).
(ii) generalized interior of A is defined as the union of all g-open subsets of A and is denoted by Int*(A).

Definition 2.3: A subset A of a topological space (X, τ) is called
(i) semi-open [6] (resp. semi*-open[12]) if A ⊆ Cl(Int(A)) (resp. A ⊆ Cl*(Int(A)).

The class of all semi*-open (resp. semi*-closed) sets is denoted by S*O(X, τ) (resp. S*C(X,τ)).

The semi*-interior of A is defined as the union of all semi*-open sets of X contained in A. It is denoted by s*Int(A). The semi*-closure of A is defined as the intersection of all semi*-closed sets in X containing A. It is denoted by s*Cl(A).

Theorem 2.4[13]: Let A⊆X and let x∈X. Then x∈s*Cl(A) if and only if every semi*-open set in X containing x intersects A.
Theorem 2.5[12]:
(i) Every open set is semi*-open.
(ii) Every semi*-open set is semi-open.

Definition 2.6: A space X is said to be $T_i[17]$ if for every pair of distinct points x and y in X, there is an open set U containing x but not y and an open set V containing y but not x.

Definition 2.7: A space X is $R_0$ [16] if every open set contains the closure of each of its points.

Theorem 2.8:
(i) X is $R_0$, if and only if for every closed set F, $Cl({x})\cap F=\emptyset$, for all $x\in X\setminus F$.
(ii) X is semi*-R$_0$ if and only if for every semi*-closed set F, $s^*Cl({x})\cap F=\emptyset$, for all $x\in X\setminus F$.

Definition 2.9: A topological space X is said to be
(i) regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint open sets U and V in X containing x and B respectively.[17]
(ii) s-regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.[8]
(iii) semi-regular if for every pair consisting of a point x and a semi-closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.[3]

Definition 2.10: A topological space X is said to be
(i) normal if for every pair of disjoint closed sets A and B in X, there are disjoint open sets U and V in X containing A and B respectively.[17]
(ii) s-normal if for every pair of disjoint closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.[9]
(iii) semi-normal if for every pair of disjoint semi-closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.[4]

Definition 2.11: A function $f:X\rightarrow Y$ is said to be
(i) closed [17] if $f(V)$ is closed in Y for every closed set V in X.
(ii) semi*-continuous [14] if $f^{-1}(V)$ is semi*-open in X for every open set V in Y.
(iii) semi*-irresolute [15] if $f^{-1}(V)$ is semi*-open in X for every semi*-open set V in Y.
(iv) contra-semi*-irresolute [15] if $f^{-1}(V)$ is semi*-closed in X for every semi*-open set V in Y.
(v) semi*-open [14] if $f(V)$ is semi*-open in Y for every open set V in X.
(vi) pre-semi*-open [14] if $f(V)$ is semi*-open in Y for every semi*-open set V in X.
(vii) contra-pre-semi*-open [14] if $f(V)$ is semi*-closed in Y for every semi*-open set V in X.
(viii) pre-semi*-closed [14] if $f(V)$ is semi*-closed in Y for every semi*-closed set V in X.

Lemma 2.12[10]: If A and B are subsets of X such that $A\cap B=\emptyset$ and A is semi*-open in X, then $A\cap x^*Cl(B)=\emptyset$.

Theorem 2.13[15]: A function $f:X\rightarrow Y$ is semi*-irresolute if $f^{-1}(F)$ is semi*-closed in X for every semi*-closed set F in Y.

III. REGULAR SPACES ASSOCIATED WITH SEMI*-OPEN SETS.

In this section we introduce the concepts of semi*-regular and s*-regular spaces. Also we investigate their basic properties and study their relationship with already existing concepts.

Definition 3.1: A space X is said to be semi*-regular if for every pair consisting of a point x and a semi*-closed set B not containing x, there are disjoint semi*-open sets U and V in X containing x and B respectively.
Theorem 3.2: In a topological space X, the following are equivalent:

(i) X is semi*-regular.
(ii) For every \( x \in X \) and every semi*-open set U containing x, there exists a semi*-open set V containing x such that \( s^*\text{Cl}(V) \subseteq U \).
(iii) For every set A and a semi*-open set B such that \( A \cap B \neq \emptyset \), there exists a semi*-open set \( U \) such that \( A \cap U \neq \emptyset \) and \( s^*\text{Cl}(U) \subseteq B \).
(iv) For every non-empty set A and semi*-closed set B such that \( A \cap B = \emptyset \), there exist disjoint semi*-open sets U and V such that \( A \cap U \neq \emptyset \) and \( B \subseteq V \).

Proof: (i)⇒(ii): Let U be a semi*-open set containing x. Then \( B= XuU \) is a semi*-*closed not containing x. Since X is semi*-regular, there exist disjoint semi*-open sets V and W containing x and B respectively. If \( y \notin B \), W is a semi*-open set containing y that does not intersect V and hence by Theorem 2.4, y cannot belong to \( s^*\text{Cl}(V) \). Therefore \( s^*\text{Cl}(V) \subseteq U \).

(ii)⇒(iii): Let \( A \cap B \neq \emptyset \) and B be semi*-open. Let \( x \in A \cap B \). Then by assumption, there exists a semi*-open set U containing x such that \( s^*\text{Cl}(U) \subseteq B \). Since \( x \in A \), \( A \cap U \neq \emptyset \). This proves (iii).

(iii)⇒(iv): Suppose \( A \cap B = \emptyset \), where A is non-empty and B is semi*-closed. Then \( X \setminus B \) is semi*-open and \( A \cap (X \setminus B) = \emptyset \). By (iii), there exists a semi*-open set \( U \) such that \( A \cap U \neq \emptyset \) and \( U \subseteq s^*\text{Cl}(U) \setminus X \setminus B \). Put \( V = X \setminus s^*\text{Cl}(U) \). Hence \( V \) is a semi*-open set containing B such that \( U \cap V = U \cap (X \setminus s^*\text{Cl}(U)) \subseteq U \cap (X \setminus U) = \emptyset \). This proves (iv).

(iv)⇒(i): Let B be semi*-closed and \( x \notin B \). Take \( A = \{ x \} \). Then \( A \cap B = \emptyset \). By (iv), there exist disjoint semi*-open sets U and V such that \( U \cap A \neq \emptyset \) and \( B \subseteq V \). Since \( U \cap A \neq \emptyset \), \( x \in U \). This proves that X is semi*-regular.

Theorem 3.3: Let X be a semi*-regular space.

(i) Every semi*-open set in X is a union of semi*-closed sets.
(ii) Every semi*-closed set in X is an intersection of semi*-open sets.

Proof: (i) Suppose X is s*-regular. Let \( G \) be a semi*-open set and \( x \in G \). Then \( F=XuG \) is semi*-closed and \( x \notin F \). Since X is semi*-regular, there exist disjoint semi*-open sets \( U_x \) and \( V \) in X such that \( x \in U_x \) and \( F \subseteq V \). Since \( U_x \cap F \subseteq U_x \cap V = \emptyset \), we have \( U_x \subseteq XF = G \). Take \( V_x = s^*\text{Cl}(U_x) \). Then \( V_x \) is semi*-closed and by Lemma 2.12, \( V_x \cap V = \emptyset \). Now \( F \subseteq V \) implies that \( V_x \cap F \subseteq V_x \cap V = \emptyset \). It follows that \( x \in V_x \subseteq XF = G \). This proves that \( G = \bigcup \{ V_x : x \in G \} \). Thus G is a union of semi*-closed sets.

(ii) Follows from (i) and set theoretic properties.

Theorem 3.4: If \( f \) is a semi*-irresolute and pre-semi*-closed injection of a topological space X into a semi*-regular space Y, then X is semi*-regular.

Proof: Let \( x \in X \) and \( A \) be a semi*-closed set in X not containing x. Since \( f \) is pre-semi*-closed, \( f(A) = f(\{ x \}) \) is a semi*-closed set in Y not containing \( f(x) \). Since Y is semi*-regular, there exist disjoint semi*-open sets \( V_1 \) and \( V_2 \) in Y such that \( f(x) \in V_1 \) and \( f(A) \subseteq V_2 \). Since \( f \) is semi*-irresolute, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are disjoint semi*-open sets in X containing x and A respectively. Hence X is semi*-regular.

Theorem 3.5: If \( f \) is a semi*-continuous and closed injection of a topological space X into a regular space Y and if every semi*-closed set in X is closed, then X is semi*-regular.
Proof: Let \( x \in X \) and \( A \) be a semi*-closed set in \( X \) not containing \( x \). Then by assumption, \( A \) is closed in \( X \). Since \( f \) is closed, \( f(A) \) is a closed set in \( Y \) not containing \( f(x) \). Since \( Y \) is regular, there exist disjoint open sets \( V_1 \) and \( V_2 \) in \( Y \) such that \( f(x) \in V_1 \) and \( f(A) \subseteq V_2 \). Since \( f \) is semi*-continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are disjoint semi*-open sets in \( X \) containing \( x \) and \( A \) respectively. Hence \( X \) is semi*-regular.

Theorem 3.6: If \( f : X \rightarrow Y \) is a semi*-irresolute bijection which is pre-semi*-open and \( X \) is semi*-regular. Then \( Y \) is also semi*-regular.

Proof: Let \( f : X \rightarrow Y \) be a semi*-irresolute bijection which is semi*-open and \( X \) be semi*-regular. Let \( y \in Y \) and \( B \) be a semi*-closed set in \( Y \) not containing \( y \). Since \( f \) is semi*-irresolute, by Theorem 2.13 \( f^{-1}(B) \) is a semi*-closed set in \( X \) not containing \( f^{-1}(y) \). Since \( X \) is semi*-regular, there exist disjoint semi*-open sets \( U_1 \) and \( U_2 \) containing \( f^{-1}(y) \) and \( f^{-1}(B) \) respectively. Since \( f \) is pre-semi*-open, \( f(U_1) \) and \( f(U_2) \) are disjoint semi*-open sets in \( Y \) containing \( y \) and \( B \) respectively. Hence \( Y \) is semi*-regular.

Theorem 3.7: If \( f \) is a continuous semi*-open bijection of a regular space \( X \) into a space \( Y \) and if every semi*-closed set in \( Y \) is closed, then \( Y \) is semi*-regular.

Proof: Let \( y \in Y \) and \( B \) be a semi*-closed set in \( Y \) not containing \( y \). Then by assumption, \( B \) is closed in \( Y \). Since \( f \) is a continuous bijection, \( f^{-1}(B) \) is a closed set in \( X \) not containing the point \( f^{-1}(y) \). Since \( X \) is regular, there exist disjoint open sets \( U_1 \) and \( U_2 \) in \( X \) such that \( f^{-1}(y) \in U_1 \) and \( f^{-1}(B) \subseteq U_2 \). Since \( f \) is semi*-open, \( f(U_1) \) and \( f(U_2) \) are disjoint semi*-open sets in \( Y \) containing \( y \) and \( B \) respectively. Hence \( Y \) is semi*-regular.

Theorem 3.8: If \( X \) is semi*-regular, then it is semi-R_0.

Proof: Suppose \( X \) is semi*-regular. Let \( U \) be a semi*-open set and \( x \in U \). Take \( F = X \setminus U \). Then \( F \) is a semi*-closed set not containing \( x \). By semi*-regularity of \( X \), there are disjoint semi*-open sets \( V \) and \( W \) such that \( x \in V \), \( F \subseteq W \). If \( y \in F \), then \( W \) is a semi*-open set containing \( y \) that does not intersect \( V \). Therefore \( y \notin s^*Cl(\{x\}) \Rightarrow y \notin s^*Cl(\{x\}) \cap V = \emptyset \) and hence \( s^*Cl(\{x\}) \subseteq X \setminus F = U \). Hence \( X \) is semi-R_0.

Definition 3.9: A space \( X \) is said to be \( s^*-\text{regular} \) if for every pair consisting of a point \( x \) and a closed set \( B \) not containing \( x \), there are disjoint semi*-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( B \) respectively.

Theorem 3.10: (i) Every regular space is \( s^*-\text{regular} \).
(ii) Every \( s^*-\text{regular} \) space is \( s\)-regular.

Proof: Suppose \( X \) is regular. Let \( F \) be a closed set and \( x \notin F \). Since \( X \) is regular, there exist disjoint open sets \( U \) and \( V \) containing \( x \) and \( F \) respectively. Then by Theorem 2.5(i) \( U \) and \( V \) are semi*-open in \( X \). This implies that \( X \) is \( s^*-\text{regular} \). This proves (i).

Suppose \( X \) is \( s^*-\text{regular} \). Let \( F \) be a closed set and \( x \notin F \). Since \( X \) is \( s^*-\text{regular} \), there exist disjoint semi*-open sets \( U \) and \( V \) containing \( x \) and \( F \) respectively. Then by Theorem 2.5(ii) \( U \) and \( V \) are semi-open in \( X \). This implies that \( X \) is \( s\)-regular. This proves (ii).

Remark 3.11: The reverse implications of the statements in the above theorem are not true as shown in the following examples.

Example 3.12: Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\} \). Clearly \((X, \tau)\) is \( s^*-\text{regular} \) but not regular.

Example 3.13: Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \). Clearly \((X, \tau)\) is \( s\)-regular but not \( s^*-\text{regular} \).

Theorem 3.14: For a topological space \( X \), the following are equivalent:
(i) X is $s^*$-regular.
(ii) For every $x \in X$ and every open set $U$ containing $x$, there exists a $s^*$-open set $V$ containing $x$ such that $s^*\text{Cl}(V) \subseteq U$.
(iii) For every set $A$ and an open set $B$ such that $A \cap B \neq \emptyset$, there exists a $s^*$-open set $U$ such that $A \cap U \neq \emptyset$ and $s^*\text{Cl}(U) \subseteq B$.
(iv) For every non-empty set $A$ and closed set $B$ such that $A \cap B = \emptyset$, there exist disjoint $s^*$-open sets $U$ and $V$ such that $A \cap U \neq \emptyset$ and $B \subseteq V$.

**Proof:** (i)$\Rightarrow$(ii): Let $U$ be an open set containing $x$. Then $B = X \setminus U$ is a closed set not containing $x$. Since $X$ is $s^*$-regular, there exist disjoint $s^*$-open sets $V$ and $W$ containing $x$ and $B$ respectively. If $y \in B$, $W$ is a $s^*$-open set containing $y$ that does not intersect $V$ and hence by Theorem 3.1, $y$ cannot belong to $s^*\text{Cl}(V)$. Therefore $s^*\text{Cl}(V)$ is disjoint from $B$. Hence $s^*\text{Cl}(V) \subseteq U$.
(ii)$\Rightarrow$(iii): Let $A \cap B \neq \emptyset$ and $B$ be open. Let $x \in A \cap B$. Then by assumption, there exists a $s^*$-open set $U$ containing $x$ such that $s^*\text{Cl}(U) \subseteq B$. Since $x \in A$, $A \cap U \neq \emptyset$. This proves (iii).
(iii)$\Rightarrow$(iv): Suppose $A \cap B = \emptyset$, where $A$ is non-empty and $B$ is closed. Then $X \setminus B$ is open and $A \cap (X \setminus B) \neq \emptyset$. By (iii), there exists a $s^*$-open set $U$ such that $A \cap U \neq \emptyset$, and $U \subseteq s^*\text{Cl}(U) \subseteq X \setminus B$. Put $V = X \setminus s^*\text{Cl}(U)$. Hence $V$ is a $s^*$-open set containing $B$ such that $U \cap V = U \cap (X \setminus s^*\text{Cl}(U)) \subseteq U \cap \emptyset = \emptyset$. This proves (iv).
(iv)$\Rightarrow$(i): Let $B$ be closed and $x \notin B$. Take $A = \{x\}$. Then $A \cap B = \emptyset$. By (iv), there exist disjoint $s^*$-open sets $U$ and $V$ such that $U \cap A \neq \emptyset$ and $B \subseteq V$. Since $U \cap A \neq \emptyset$, $x \in U$. This proves that $X$ is $s^*$-regular.

**Theorem 3.15:** If $X$ is a regular $T_1$ space, then for every pair of distinct points of $X$ there exist $s^*$-open sets containing them whose $s^*$-closures are disjoint.

**Proof:** Let $x$, $y$ be two distinct points in the regular $T_1$ space $X$. Since $X$ is $T_1$, $\{y\}$ is closed. Since $X$ is regular, there exist disjoint open sets $U_1$ and $U_2$ containing $x$ and $\{y\}$ respectively. By Theorem 3.9(i), $X$ is $s^*$-regular and hence by Theorem 3.13, there exist $s^*$-open sets $V_1$ and $V_2$ containing $x$, $y$ such that $s^*\text{Cl}(V_1) \subseteq U_1$ and $s^*\text{Cl}(V_2) \subseteq U_2$. Since $U_1$ and $U_2$ are disjoint, $s^*\text{Cl}(V_1)$ and $s^*\text{Cl}(V_2)$ are disjoint. This proves the theorem.

**Theorem 3.16:** Every $s^*$-regular space is $s^*$-regular.

**Proof:** Suppose $X$ is semi*$^*$-regular. Let $F$ be a closed set and $x \in F$. Then by Theorem (i), $F$ is semi*-closed in $X$. Since $X$ is semi*$^*$-regular, there exist disjoint semi*$^*$-open sets $U$ and $V$ containing $x$ and $F$ respectively. This implies that $X$ is $s^*$-regular.

**Theorem 3.17:** (i) Every $s^*$-regular $T_1$ space is semi*$^*$-$T_2$.
(ii) Every semi*$^*$-regular semi*$^*$-$T_1$ space is semi*$^*$-$T_2$.

**Proof:** Suppose $X$ is $s^*$-regular and $T_1$. Let $x$ and $y$ be two distinct points in $X$. Since $X$ is $T_1$, $\{x\}$ is closed and $y \notin \{x\}$. Since $X$ is $s^*$-regular, there exist disjoint $s^*$-open sets $U$ and $V$ in $X$ containing $\{x\}$ and $y$ respectively. It follows that $X$ is semi*$^*$-$T_2$. This proves (i).
Suppose $X$ is semi*$^*$-regular and semi*$^*$-$T_1$. Let $x$ and $y$ be two distinct points in $X$. Since $X$ is semi*$^*$-$T_1$, $\{x\}$ is semi*-closed and $y \notin \{x\}$. Since $X$ is semi*$^*$-regular, there exist disjoint semi*-open sets $U$ and $V$ in $X$ containing $\{x\}$ and $y$ respectively. It follows that $X$ is semi*$^*$-$T_2$. This proves (ii).

**Theorem 3.18:** Let $X$ be an $s^*$-regular space.

i) Every open set in $X$ is a union of semi*-closed sets.
Every closed set in $X$ is an intersection of semi-open sets.

**Proof:** (i) Suppose $X$ is $s$-regular. Let $G$ be an open set and $x \in G$. Then $F=X\setminus G$ is closed and $x \notin F$. Since $X$ is $s$-regular, there exist disjoint semi-open sets $U_x$ and $U$ in $X$ such that $x \in U_x$ and $F \subseteq U$. Since $U_x \cap F \subseteq U_x \cap U = \emptyset$, we have $U_x \subseteq X \setminus F = G$. Take $V_x = s^*\text{Cl}(U_x)$. Then $V_x$ is semi-closed. Now $F \subseteq U$ implies that $V_x \cap F \subseteq V_x \cap U = \emptyset$. It follows that $x \in V_x \subseteq X \setminus F = G$. This proves that $G = \bigcup \{ V_x : x \in G \}$. Thus $G$ is a union of semi-closed sets.

(ii) Follows from (i) and set theoretic properties.

**IV. NORMAL SPACES ASSOCIATED WITH SEMI*-OPEN SETS.**

In this section we introduce variants of normal spaces namely semi-normal spaces and $s^*$-normal spaces and investigate their basic properties. We also give characterizations for these spaces.

**Definition 4.1:** A space $X$ is said to be **semi-normal** if for every pair of disjoint semi-closed sets $A$ and $B$ in $X$, there are disjoint semi-open sets $U$ and $V$ in $X$ containing $A$ and $B$ respectively.

**Theorem 4.2:** In a topological space $X$, the following are equivalent:

(i) $X$ is semi-normal.

(ii) For every semi-closed set $A$ in $X$ and every semi-open set $U$ containing $A$, there exists a semi-open set $V$ containing $A$ such that $s^*\text{Cl}(V) \subseteq U$.

(iii) For each pair of disjoint semi-closed sets $A$ and $B$ in $X$, there exists a semi-open set $U$ containing $A$ such that $s^*\text{Cl}(U) \cap B = \emptyset$.

(iv) For each pair of disjoint semi-closed sets $A$ and $B$ in $X$, there exist semi-open sets $U$ and $V$ containing $A$ and $B$ respectively such that $s^*\text{Cl}(U) \cap s^*\text{Cl}(V) = \emptyset$.

**Proof:**

(i)$\Rightarrow$(ii): Let $U$ be a semi-open set containing the semi-closed set $A$. Then $B=X\setminus U$ is a semi-closed set disjoint from $A$. Since $X$ is semi-normal, there exist disjoint semi-open sets $V$ and $W$ containing $A$ and $B$ respectively. Then $s^*\text{Cl}(V)$ is disjoint from $B$, since if $y \in B$, the set $W$ is a semi-open set containing $y$ disjoint from $V$. Hence $s^*\text{Cl}(V) \subseteq U$.

(ii)$\Rightarrow$(iii): Let $A$ and $B$ be disjoint semi-closed sets in $X$. Then $X \setminus B$ is a semi-open set containing $A$. By (ii), there exists a semi-open set $U$ containing $A$ such that $s^*\text{Cl}(U) \subseteq X \setminus B$. Hence $s^*\text{Cl}(U) \cap B = \emptyset$. This proves (iii).

(iii)$\Rightarrow$(iv): Let $A$ and $B$ be disjoint semi-closed sets in $X$. Then, by (iii), there exists a semi-open set $U$ containing $A$ such that $s^*\text{Cl}(U) \cap B = \emptyset$. Since $s^*\text{Cl}(U)$ is semi-closed, $B$ and $s^*\text{Cl}(U)$ are disjoint semi-closed sets in $X$. Again by (iii), there exists a semi-open set $V$ containing $B$ such that $s^*\text{Cl}(U) \cap s^*\text{Cl}(V) = \emptyset$. This proves (iv).

(iv)$\Rightarrow$(i): Let $A$ and $B$ be the disjoint semi-closed sets in $X$. By (iv), there exist semi-open sets $U$ and $V$ containing $A$ and $B$ respectively such that $s^*\text{Cl}(U) \cap s^*\text{Cl}(V) = \emptyset$. Since $U \cap V \subseteq s^*\text{Cl}(U) \cap s^*\text{Cl}(V)$, $U$ and $V$ are disjoint semi-open sets containing $A$ and $B$ respectively. Thus $X$ is semi-normal.

**Theorem 4.3:** For a space $X$, the following are equivalent:

(i) $X$ is semi-normal.

(ii) For any two semi-open sets $U$ and $V$ whose union is $X$, there exist semi-closed subsets $A$ of $U$ and $B$ of $V$ whose union is also $X$.
Proof: (i)⇒(ii): Let U and V be two semi*-open sets in a semi*-normal space X such that X=U∪V. Then X∪, X∪V are disjoint semi*-closed sets. Since X is semi*-normal, there exist disjoint semi*-open sets G₁ and G₂ such that X∪₁∪₁ograms and X∪V₁∪₂ emphas. Let A=X∪₁ and B=X∪₂. Then A and B are semi*-closed subsets of U and V respectively such that A∪B=X. This proves (ii).

(ii)⇒(i): Let A and B be disjoint semi*-closed sets in X. Then X\A and X\B are semi*-open sets whose union is X. By (ii), there exists semi*-closed sets F₁ and F₂ such that F₁⊆X\A, F₂⊆X\B and F₁∪F₂=X. Then X\F₁ and X\F₂ are disjoint semi*-open sets containing A and B respectively. Therefore X is semi*-normal.

Definition 4.4: A space X is said to be s**-normal if for every pair of disjoint closed sets A and B in X, there are disjoint semi*-open sets U and V in X containing A and B respectively.

Theorem 4.5: (i) Every normal space is s**-normal.
(ii) Every s**-normal space is s-normal.
(iii) Every semi*-normal space is s**-normal.

Proof: Suppose X is normal. Let A and B be disjoint closed sets in X. Since X is normal, there exist disjoint open sets U and V containing A and B respectively. Then by Theorem 2.5(i), U and V are semi*-open in X. This implies that X is s**-normal. This proves (i).

Suppose X is s**-normal. Let A and B be disjoint closed sets in X. Since X is s**-normal, there exist disjoint semi*-open sets U and V containing A and B respectively. Then by Theorem 2.5(ii), U and V are semi-open in X. This implies that X is s-normal. This proves (ii).

Suppose X is semi*-regular. Let A and B be disjoint closed sets in X. Then by Theorem 2.5(i), A and B are disjoint semi*-closed sets in X. Since X is semi*-regular, there exist disjoint semi*-open sets U and V containing A and B respectively. Therefore X is s**-normal. This proves (iii).

Theorem 4.6: In a topological space X, the following are equivalent:
(i) X is s**-normal.
(ii) For every closed set F in X and every open set U containing F, there exists a semi*-open set V containing F such that s*Cl(V)⊆U.
(iii) For each pair of disjoint closed sets A and B in X, there exists a semi*-open set U containing A such that s*Cl(U)∩B=∅.

Proof: (i)⇒(ii): Let U be a open set containing the closed set F. Then H=X\U is a closed set disjoint from F. Since X is s**-normal, there exist disjoint semi*-open sets V and W containing F and H respectively. Then s*Cl(V) is disjoint from H, since if y∈H, the set W is a semi*-open set containing y disjoint from V. Hence s*Cl(V)⊆U.

(ii)⇒(iii): Let A and B be disjoint closed sets in X. Then X\B is an open set containing A. By (ii), there exists a semi*-open set U containing A such that s*Cl(U)⊆ X\B. Hence s*Cl(U)∩B=∅. This proves (iii).

(iii)⇒(i): Let A and B be the disjoint semi*-closed sets in X. By (iii), there exists a semi*-open set U containing A such that s*Cl(U)∩B=∅. Take V=X\s*Cl(U). Then U and V are disjoint semi*-open sets containing A and B respectively. Thus X is s**-normal.

Theorem 4.7: For a space X, then the following are equivalent:
(i) X is s**-normal.
(ii) For any two open sets $U$ and $V$ whose union is $X$, there exist semi*-closed subsets $A$ of $U$ and $B$ of $V$ whose union is also $X$.

**Proof:** (i)⇒(ii): Let $U$ and $V$ be two open sets in an $s^{**}$-normal space $X$ such that $X=U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint closed sets. Since $X$ is $s^{**}$-normal, there exist disjoint semi*-open sets $G_1$ and $G_2$ such that $X \setminus U \subseteq G_1$ and $X \setminus V \subseteq G_2$. Let $A=X \cap G_1$ and $B=X \cap G_2$. Then $A$ and $B$ are semi*-closed subsets of $U$ and $V$ respectively such that $A \cup B=X$. This proves (ii).

(ii)⇒(i): Let $A$ and $B$ be disjoint closed sets in $X$. Then $X \setminus A$ and $X \setminus B$ are open sets whose union is $X$. By (ii), there exists semi*-closed sets $F_1$ and $F_2$ such that $F_1 \subseteq X \setminus A$, $F_2 \subseteq X \setminus B$ and $F_1 \cup F_2=X$. Then $X \setminus F_1$ and $X \setminus F_2$ are disjoint semi*-open sets containing $A$ and $B$ respectively. Therefore $X$ is $s^{**}$-normal.

**Remark 4.8:** It is not always true that an $s^{**}$-normal space $X$ is $s^*$-regular as shown in the following example. However it is true if $X$ is $R_0$ as seen in Theorem 4.10.

**Example 4.9:** Let $X=\{a, b, c, d\}$ with topology $\tau=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Clearly $(X, \tau)$ is $s^{**}$-normal but not $s^*$-regular.

**Theorem 4.10:** Every $s^{**}$-normal $R_0$ space is $s^*$-regular.

**Proof:** Suppose $X$ is $s^{**}$-normal and $R_0$. Let $F$ be a closed set and $x \notin F$. Since $X$ is $R_0$, by Theorem 2.8(i), $Cl(\{x\}) \cap F = \emptyset$. Since $X$ is $s^{**}$-normal, there exist disjoint semi*-open sets $U$ and $V$ in $X$ containing $Cl(\{x\})$ and $F$ respectively. It follows that $X$ is $s^*$-regular.

**Corollary 4.11:** Every $s^{**}$-normal $T_1$ space is $s^*$-regular.

**Proof:** Follows from the fact that every $T_1$ space is $R_0$ and Theorem 4.10.

**Theorem 4.12:** If $f$ is an injective and semi*-irresolute and pre-semi*-closed mapping of a topological space $X$ into a semi*-normal space $Y$, then $X$ is semi*-normal.

**Proof:** Let $f$ be an injective and semi*-irresolute and pre-semi*-closed mapping of a topological space $X$ into a semi*-normal space $Y$. Let $A$ and $B$ be disjoint semi*-closed sets in $X$. Since $f$ is a pre-semi*-closed function, $f(A)$ and $f(B)$ are disjoint semi*-closed sets in $Y$. Since $Y$ is semi*-normal, there exist disjoint semi*-open sets $V_1$ and $V_2$ in $Y$ containing $f(A)$ and $f(B)$ respectively. Since $f$ is semi*-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint semi*-open sets in $X$ containing $A$ and $B$ respectively. Hence $X$ is semi*-normal.

**Theorem 4.13:** If $f$ is an injective and semi*-continuous and closed mapping of a topological space $X$ into a normal space $Y$ and if every semi*-closed set in $X$ is closed, then $X$ is semi*-normal.

**Proof:** Let $A$ and $B$ be disjoint semi*-closed sets in $X$. By assumption, $A$ and $B$ are closed in $X$. Then $f(A)$ and $f(B)$ are disjoint closed sets in $Y$. Since $Y$ is normal, there exist disjoint open sets $V_1$ and $V_2$ in $Y$ such that $f(A) \subseteq V_1$ and $f(B) \subseteq V_2$. Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint semi*-open sets in $X$ containing $A$ and $B$ respectively. Hence $X$ is semi*-normal.

**Theorem 4.14:** If $f:X \to Y$ is a semi*-irresolute injection which is pre-semi*-open and $X$ is semi*-normal, then $Y$ is also semi*-normal.

**Proof:** Let $f:X \to Y$ be a semi*-irresolute surjection which is semi*-open and $X$ be semi*-normal. Let $A$ and $B$ be disjoint semi*-closed sets in $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint semi*-closed sets in $X$. Since $X$ is semi*-normal, there exist disjoint semi*-open sets $U_1$ and $U_2$ containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Since $f$ is pre-semi*-open, $f(U_1)$ and $f(U_2)$ are disjoint semi*-open sets in $Y$ containing $A$ and $B$ respectively. Hence $Y$ is semi*-normal.
Remark 4.15: It is not always true that a semi*-normal space $X$ is semi*-regular as shown in the following example. However it is true if $X$ is semi*-R$_0$ as seen in Theorem 4.17.

Example 4.16: Let $X=\{a, b, c, d\}$ with topology $\tau=\{\emptyset, \{a, b\}, X\}$. Clearly $(X, \tau)$ is semi*-normal but not semi*-regular.

Theorem 4.17: Every semi*-normal space that is semi*-R$_0$ is semi*-regular.

Proof: Suppose $X$ is semi*-normal that is semi*-R$_0$. Let $F$ be a semi*-closed set and $x \notin F$. Since $X$ is semi*-R$_0$, by Theorem 2.8(ii), $s^*\text{Cl}(\{x\}) \cap F = \emptyset$. Since $X$ is semi*-normal, there exist disjoint semi*-open sets $U$ and $V$ in $X$ containing $s^*\text{Cl}(\{x\})$ and $F$ respectively. It follows that $X$ is semi*-regular.

Corollary 4.18: Every semi*-normal semi*-T$_1$ space is semi*-regular.

Proof: Follows from the fact that every semi*-T$_1$ space is semi*-R$_0$ and Theorem 4.13.

REFERENCES