Research Inventy: International Journal Of Engineering And Science
Vol.4, Issue 1 (January 2014), PP 46-51
Issn(e): 2278-4721, Issn(p):2319-6483, www.researchinventy.com

Numerical Solution of Second Order Nonlinear Fredholm-Volterra Integro Differential Equations by Canonical Basis Function

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ABSTRACT : *This paper deals with the construction of Canonical Polynomials basis function and used to find approximation solutions of Second order nonlinear Fredholm – Volterra Integro Differential Equations. The solutions obtained are compared favorably with the solutions obtained by Cerdik-Yaslan et. al [1]. One of the advantages of the method discussed is that solution is expressed as a truncated Canonicals series, then both the exact and the approximate solutions are easily evaluated for arbitrary values of x in the intervals of consideration to obtain numerical values for both solutions. Finally, some examples of second order nonlinear Fredholm – Volterra Integro Differential Equations are presented to illustrate the method.*

KEY WORDS: *Canonical Polynomials, Nonlinear Fredholm – Volterra integro Differential Equations, Approximate solution*

I. INTRODUCTION

In recent years, there has been a growing interest in the Integro Differential Equations (IDEs). IDEs play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics. Higher order integral differential equations arise in mathematical, applied and engineering sciences, astrophysics, solid state physics, astronomy, fluid dynamics, beam theory etc. to mention a few are usually very difficulty to solve analytically, so numerical method is required. Variational Iteration Method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [2,3,4,5]. VIM has successfully been applied to many situations. For example, He [4] solved the classical Blasius equation using VIM. He [2] used VIM to give approximate solutions for some well known nonlinear problems. He [3] used VIM to solve the well known Blasius equation. He [5] solved strongly nonlinear equation using VIM. Taiwo [6] used Canonical Polynomials to solve Singularly Perturbed Boundary Value Problems. The idea reported in Taiwo [6] motivated the work reported in this paper in which Canonical Polynomials obtained for Singularly Perturbed Boundary Value Problems are reformulated and used to solve nonlinear Fredholm – Volterra Integro Differential Equations. Some of the advantages of Canonical Polynomials constructed are it is generated recursively, it could be converted to any interval of consideration and it is easy to use and user's friendly in term of computer implementation.

II. GENERAL NONLINEAR FREDHOLM – VOLTERRA INTEGRO DIFFERENTIAL EQUATION CONSIDERED

The general nonlinear Fredholm – Volterra Differential Equation considered in the paper is of the form:

$$\sum_{k=0}^{m} P_k y^k(x) = g(x) + \lambda_1 \int_{-1}^{1} F(x,t)y(t)dt + \lambda_2 \int_{-1}^{x} K(x,t)y(t)dt \tag{1}$$

With the mixed conditions

$$\sum_{j=0}^{m-1} [\, a_{ij} y^{(j)}(-1) + b_{ij} y^{(j)}(1) + c_{ij} y^{(j)}(c)\,] = \alpha_i \quad i = 0(1)n \tag{2}$$

Where $y(x)$ *and* $y(t)$ are unknown functions, $g(x)$, $K(x,t)$, $F(x,t)$ are smooth functions, P_k, a_{ij}, b_{ij}, c_{ij}, λ_1, λ_2, α_i are constants or transcendental or hyperbolic functions and $P_0 \neq 0$

III CONSTRUCTION OF CANONICAL POLYNOMIALS

The construction of Canonical Polynomials is carried out by expanding the left hand side of (1). Thus from (1), we write the left hand side in an operator form as:

$$Ly = P_0 + P_1 \frac{d}{dx} + P_2 \frac{d^2}{dx^2} + \quad ... + P_m \frac{d^m}{dx^m}$$

Let,

$$LQ_r(x) = x^k \tag{3}$$

Then,

$$Lx^k = P_0 x^k + P_1 k x^{k-1} + P_2 k(k-1)x^{k-2} + P_3 k(k-1)(k-2)x^{k-3} + ... \tag{4}$$

Thus, making use of (3), we have

$$L[LQ_k(x)] = P_0 LQ_k(x) + P_1 k LQ_{k-1}(x) + P_2\, k(k-1)LQ_{k-2}(x) \tag{5}$$

Hence, making use of (3) in (5), we have

$$Q_k(x) = \frac{1}{P_0}[\, x^k - P_1 k Q_{k-1}(x) - P_2 k\,(k-1)Q_{k-2}(x) + ... \,] \tag{6}$$

Thus, (6) is the recurrence relation of the Canonical Polynomials.

IV. DESCRIPTION OF STANDARD METHOD BY CANONICAL POLYNOMIALS

In this section, the Standard method by Canonical Polynomials for solving (1) and (2) is discussed. The method assumed an approximate solution of the form:

$$y(x) \equiv y_n(x) = \sum_{r=0}^{n} a_r Q_r(x) \tag{7}$$

Where n is the degree of approximant, $a_r\ (r \ge 0)$ are constants to be determined and $Q_r(x)$ are the Canonical Polynomials generated recursively by (6).

Substituting (7) into (1), we obtain

$$P_0 \sum_{r=0}^{n} a_r Q_r(x) + P_1 \sum_{r=0}^{n} a_r Q'(x) + P_2 \sum_{r=0}^{n} a_r Q_r''(x) + ...+$$
$$= g(x) + \lambda_1 \int_{-1}^{1} F(x,t) \sum_{r=0}^{n} a_r Q_r(t) dt + \lambda_2 \int_{-1}^{x} K(x,t) \sum_{r=0}^{n} a_r Q_r(t) dt \tag{8}$$

This is further simplified to give

$$P_O\{a_o Q_O(x) + a_1 Q_1(x) + a_2 Q_2(x) + a_3 Q_3 + ...\}$$
$$+ P_1\{a_o Q'_O(x) + a_1 Q'_1(x) + a_2 Q'_2(x) + a_3 Q'_3 + ...\}$$
$$+ P_2\{a_o Q''_O(x) + a_1 Q''_1(x) + a_2 Q''_2(x) + a_3 Q''_3 + ...\}$$
$$\vdots$$
$$\vdots$$
$$\vdots$$
$$+ P_m\{a_o Q^m{}_O(x) + a_1 Q^m{}_1(x) + a_2 Q^m{}_2(x) + a_3 Q^m{}_3 + ...\}$$
$$+ g(x) + \lambda_1 \int_{-1}^{1} F(x,t)\{a_o Q_0(t) + a_1 Q_1(t) + a_2 Q_2(t) + ... + a_N Q_N(t)\}dt$$
$$+ \lambda_2 \int_{-1}^{1} k(x,t)\{a_o Q_0(t) + a_1 Q_1(t) + a_2 Q_2(t) + ... + a_N Q_N(t)\}dt \tag{9}$$

Collect like terms in (9), we obtain

$$\left\{p_oQ_o(x)+p_1Q_o^{'}(x)+\ldots+p_mQ_o^m(x)+\lambda_1\int_{-1}^{1}F(x,t)Q_o(t)dt+\lambda_2\int_{-1}^{x}k(x,t)Q_o(t)\right\}a_0$$
$$+\left\{p_1Q_1(x)+p_1Q_1^{'}(x)+\ldots+p_mQ_1^m(x)+\lambda_1\int_{-1}^{1}F(x,t)Q_1(t)dt+\lambda_2\int_{-1}^{x}k(x,t)Q_1(t)dt\right\}a_1$$
$$+\left\{p_2Q_2(x)+p_2Q_2^{'}(x)+\ldots+p_mQ_2^m(x)+\lambda_1\int_{-1}^{1}F(x,t)Q_2(t)dt+\lambda_2\int_{-1}^{x}k(x,t)Q_2(t)dt\right\}a_2$$
$$\vdots$$
$$\vdots$$
$$\vdots$$
$$+\left\{\begin{array}{l}p_oQ_N(x)+p_1Q_N^{'}(x)+p_2Q_N(x)+\ldots+p_mQ_N^m(x)+\lambda_1\int_{-1}^{1}F(x,t)Q_N(t)dt\\+\lambda_2\int_{-1}^{x}k(x,t)Q_N(t)dt\end{array}\right\}a_N=g(x) \qquad (10)$$

Thus, (10) is collocated at point $x = x_i$, we obtain

$$\left\{p_oQ_o(x_i)+p_1Q'_o(x_i)+\ldots+p_mQ_o^m(x_i)+\lambda_1\int_{-1}^{1}F(x_i,t)Q_o(t)dt+\lambda_2\int_{-1}^{x}k(x_i,t)Q_o(t)\right\}a_0$$
$$+\left\{p_oQ_1(x_i)+p_1Q'_1(x_i)+\ldots+p_mQ_1^m(x_i)+\lambda_1\int_{-1}^{1}F(x_i,t)Q_1(t)dt+\lambda_2\int_{-1}^{x}k(x_i,t)Q_1(t)\right\}a_1$$
$$+\left\{p_oQ_2(x_i)+p_1Q_2^{'}(x_i)+\ldots+p_mQ_2^m(x_i)+\lambda_1\int_{-1}^{1}F(x_i,t)Q_2(t)dt+\lambda_2\int_{-1}^{x}k(x_i,t)Q_2(t)\right\}a_2$$
$$\vdots$$
$$\vdots$$
$$\vdots$$
$$+\left\{p_oQ_N(x_i)+p_1Q_N^{'}(x_i)+\ldots+p_mQ_N^m(x_i)+\lambda_1\int_{-1}^{1}F(x_i,t)Q_N(t)dt+\lambda_2\int_{-1}^{x}k(x_i,t)Q_N(t)\right\}a_N \qquad (11)$$

Where,

$$x_i = a+\frac{(b-a)_i}{N-m+2},\quad i=1,\ 2,\ 3,\ \ldots,\mathrm{N-m}+1 \qquad (12)$$

Thus, substituting (12) into (11) gives rise to N-m+1 algebraic linear equation in (N+1) unknown constants $a_r(r\geq 0)$. hence, extra m equations are obtained from (2). Altogether, we have (N+1) algebraic linear equation in (N+1) unknown constants. These equations are then solved by Gaussian Elimination Method to obtain the unknown constants which are then substituted into (7) to obtain the approximate solution for the value of N.

V . Demonstration of Standard Method by Canonical Basis Function on Examples

We solved the Fredholm-Volterra nonlinear Integro Differential equation of the form:

Example 1:

$$y^{''}(x)-xy'(x)+xy(x)=g(x)+\int_{-1}^{1}xty(t)dt+\int_{-1}^{x}(x-2t)y^2(t)dt \qquad (13)$$

Where

$$g(x)=\frac{2}{15}x^6-\frac{1}{3}x^4=x^3-2x^2-\frac{23}{15}x+\frac{5}{3}$$

together with the conditions

$y(0) = -1$
and,
$y'(0) = 1$

For N = 2, we seek the approximate solution $y_2(x)$ as a truncated Canonical series.

$$y_2(x) = \sum_{r=0}^{2} a_r Q_r(x); \quad -1 \leq x \leq 1$$

$$y_2(x) = a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x)$$

The values of $Q_0(x),\ Q_1(x)$ *and* $Q_2(x)$ *are given as* $1, x$ *and* $x^2 - 2$

Therefore,

$$y_2(x) = a_0 + a_1 x + a_2(x^2 - 2)$$

$$y_2'(x) = a_1 + 2a_2 x$$

$$y_2''(x) = 2a_2$$

$$y_2(t) = a_0 + a_1 t + a_2(t^2 - 2)$$

Equation (13) now becomes

$$2a_2 - x(a_1 + 2a_2 x) + x(a_0 + a_1 x + a_2(x^2 - 2)) = g(x) + \int_{-1}^{1} xt(a_0 + a_1 t + a_2(t^2 - 2))dt + \int_{-1}^{x} (x - 2t)(a_0 + a_1 t + a_2(t^2 - 2))^2\, dt \tag{14}$$

Thus, (14) is then simplified further and we obtain

$$2a_2 - x(a_1 + 2a_2 x) + x(a_0 + a_1 x + a_2(x^2 - 2)) = \frac{2}{15}x^6 - \frac{1}{2}x^4 + x^3 - 2x^2 + \frac{5}{2} + \frac{2}{3}a_1 x + a_0^2 x + a_0^2 - \frac{a_0 a_2 x^2}{3} - a_0 a_1 x - \frac{4a_0 a_1}{3} - \frac{a_0 a_2}{3}x^4 - \frac{10 a_0 a_2 x}{3} - 3a_0 a_2 - \frac{a_1^2 x^4}{6} + \frac{a_1^2 x}{3} + \frac{a_2^2}{2} - \frac{3a_1 a_2 x^5}{10} + \frac{2a_1 a_2 x^2}{3} + \frac{3a_1 a_2 x}{2} + \frac{28 a_1 a_2}{15} - \frac{2a_2^2 x^6}{15} + \frac{2a_2^2 x^4}{3} + \frac{43 a_2^2 x}{15} + \frac{7a_2^2}{3} \tag{15}$$

Comparing the powers of x in (15), we obtain the following:

For the constant terms:

$$2a_2 - a_0^2 + \frac{4a_0 a_1}{3} + 3a_0 a_2 - \frac{a_1^2}{2} - \frac{28 a_1 a_2}{15} - \frac{7a_2^2}{3} = \frac{5}{3} \tag{16}$$

For the power of x:

$$a_0 + \frac{a_1}{3} - 2a_2 - a_0^2 + a_0 a_1 + \frac{10a_0 a_2}{3} - \frac{a_1^2}{3} - \frac{3a_1 a_2}{2} - \frac{43a_2^2}{15} = -\frac{23}{15} \qquad (17)$$

For the power of x^2:

$$2a_2 + a_1 = -2 \qquad (18)$$

By using the condition equations, we obtain.

For $y_2(0) = -1$

$$a_0 - 2a_2 = -1 \qquad (19)$$

For

$y_2'(0)$:

Implies $a_1 = 1$

These equations are then solved and the results obtained are substituted into the approximate solution and after simplification gives the exact solution

$$y_2(x) = x^2 - 1$$

Example 2

We solved the Fredholm – Volterra Integro differential equation given as

$$y'(x) + xy(x) = g(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)y^2(t)dt \qquad (20)$$

Where $g(x) = \frac{-1}{30}x^6 + \frac{1}{3}x^4 + x^3 - 2x^2 - \frac{5}{3}x + \frac{4}{3}$ $\quad$ *and* $y(0) = -2$

Thus, the approximate solution given in (7) for case N=2 becomes

$$y_2(x) = \sum_{r=0}^{2} a_r Q_r(x) \qquad (21)$$

Hence,

$$y_2(x) = a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) \qquad (22)$$

For this example,

$$Q_i(x) = x^i - iQ_{i-1}(x);\ i \geq 0 \qquad (23)$$

We obtain the following

$$\begin{aligned} Q_0(x) &= 1 \\ Q_1(x) &= x - 1 \\ Q_2(x) &= x^2 - 2x + 2 \\ Q_3(x) &= x^3 - 3x^2 + 6x - 6 \end{aligned} \qquad (24)$$

Substituting, (24) into (22), we obtained

$$y_2(x) = a_0 + (x-1)a_1 + (x^2 - 2x + 2)a_2 \qquad (25)$$

Putting (25) into (20), after simplification we obtain

$$a_1 + (2x-2)a_2 + xa_0 + (x^2 - x)a_1 + (x^3 - 2x^2 + 2x)a_2 =$$

$$-\frac{1}{30}x^6+\frac{1}{3}x^4+x^3-2x^2-\frac{5}{3}+\frac{4}{3}+\int_0^1 (x+t)$$

Then simplified further, we obtain

$$a_1+(2x-2)a_2+xa_0+\left(x^2-x\right)a_1+\left(x^3-2x^2+2x\right)a_2=-\tfrac{1}{30}x^6+\frac{1}{3}x^4+x^3-2x^2$$

$$-\frac{5}{3}x+\frac{4}{3}xa_2+\frac{x^4\left\lfloor 2a_2(a_0-a_1+2a_2)+(a_1-2a_2)^2\right\rfloor}{12}+x^2\frac{(a_0-a_1+2a_2)^2}{2}$$

$$+\frac{a_2^2x^2}{30}+\frac{x^5a_2(a_1-2a_2)}{10}+x^3\frac{(a_1-2a_2)(a_0-a_1+2a_2)}{3}$$

We then collected like terms of x and the following sets of equations are obtained.

For constant term:

$$a_1-2a_2-\frac{a_0}{2}+\frac{1}{6}a_1-\frac{7}{12}a_2=\frac{4}{3}$$

For power of x:

$$2a_2+2a_0-\frac{3}{2}a_1+2a_2-\frac{4}{3}a_2=-\frac{5}{3}$$

Therefore, using the condition, we obtain

$$a_0-a_1+2a_2=-2$$

These equations are then solved and the results obtained are substituted into the approximation solution and after simplification gives the exact solution

$$y_2(x)=x^2-2$$

VI CONCLUSION

The method used in this paper is highly accurate and efficient. At lower degree of N (degree of approximant used), the method gives the exact solution for the two examples considered without extra computational cost. Also, the Canonical Polynomials generated are easily programmed and user friendly.

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