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Number of Zeros of a Polynomial in a Given Circle

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ABSTRACT: *In this paper we consider the problem of finding the number of zeros of a polynomial in a given circle when the coefficients of the polynomial or their real or imaginary parts are restricted to certain conditions. Our results in this direction generalize some known results in the theory of the distribution of zeros of polynomials.*

MATHEMATICS SUBJECT CLASSIFICATION: *30C10, 30C15*

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I. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the number of zeros of a polynomial in a given circle, Q. G. Mohammad [6] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

K. K. Dewan [2] generalized Theorem A to polynomials with complex coefficients and proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 > 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β ,

$$|\arg a_j-\beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......n$$

and

$$|a_n| \ge |a_{n-1}| \ge ..... \ge |a_1| \ge |a_0| \ .$$

Then the number f zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|} .$$

C. M. Upadhye [3] found bounds for the number of zeros of P(z) in Theorems B and C with less restrictive conditions on the coefficients , which were further improved by M. H. Gulzar [5] by proving the following resuls:

Theorem D: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β,

$$\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,\ldots\ldots n$$

and

$$k|a_n|\ge|a_{n-1}|\ge\ldots\ldots\ge|a_1|\ge\tau|a_0|\ ,$$

for $k\ge 1, 0<\tau\le 1$.Then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|+2|a_0|-\tau|a_0|(\cos\alpha-\sin\alpha+1)}{|a_0|} .$$

Theorem E: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$k\alpha_n\ge\alpha_{n-1}\ge\ldots\ldots\ge\alpha_1\ge\tau\alpha_0,$$

for $k\ge 1, 0<\tau\le 1$,then the number f zeros of P(z) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|\alpha_n|+\alpha_n)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

In this paper we find bounds for the number of zeros of a polynomial in a circle of any positive radius of which the above results are easy consequences. More precisely we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$k\alpha_n\ge\alpha_{n-1}\ge\ldots\ldots\ge\alpha_1\ge\tau\alpha_0,$$

for $k\ge 1, 0<\tau\le 1$.Then the number f zeros of P(z) in $|z|\le\frac{R}{c}\,(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0|+R\{k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

for $R \le 1$.

Remark 1: Taking $c=\dfrac{1}{\delta}$ and R=1, Theorem 1 reduces to Theorem E .

If the coefficients a_j are real i.e. $\beta_j = 0, \forall j$, we get the following result from Theorem 1:

Corollary 1: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0,$$

for $k \ge 1, 0<\tau \le 1$.Then the number f zeros of P(z) in $|z| \le \dfrac{R}{c}(R>0, c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{k(|a_n|+a_n)-\tau(|a_0|+a_0)+2|a_0|]$$

for $R \ge 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0|+R\{k(|a_n|+a_n)-\tau(|a_0|+a_0)+|a_0|]$$

for $R \le 1$.

Applying Theorem 1 to the polynomial –iP(z), we get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and

$$k\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \tau\beta_0,$$

for $k \ge 1, 0<\tau \le 1$.Then the number f zeros of P(z) in $|z| \le \dfrac{R}{c}(R>0, c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{k(|\beta_n|+\beta_n)-\tau(|\beta_0|+\beta_0)+2|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|]$$

for $R \ge 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0|+R\{k(|\beta_n|+\beta_n)-\tau(|\beta_0|+\beta_0)+|\beta_0|+|\alpha_0|+2\sum_{j=1}^{n}|\alpha_j|]$$

for $R \le 1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,\ldots\ldots n$$

and

$$k|a_n| \ge |a_{n-1}| \ge \ldots\ldots \ge |a_1| \ge \tau|a_0|\ ,$$

for $k \ge 1, 0<\tau \le 1$,then the number f zeros of P(z) in $|z| \le \dfrac{R}{c}(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[k|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|]}{|a_0|}.$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[k|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|]}{|a_0|}.$$

for $R\leq 1$.

Remark 2: Taking $c=\frac{1}{\delta}$,R=1, Theorem 3 reduces to Theorem D .

II. LEMMAS

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z|\leq R$,but not identically zero, f(0)$\neq$ 0 and $f(a_k)=0, k=1,2,......, n$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{Re}^{i\theta}|d\theta-\log|f(0)|=\sum_{j=1}^{n}\log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic and $|f(z)|\leq M(r)$ in $|z|\leq r$, then the number of zeros of f(z) in $|z|\leq\frac{r}{c}, c>1$ does not exceed

$$\frac{1}{\log c}\log\frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, 0\leq j\leq n,$ and

$|a_j|\geq|a_{j-1}|, 0\leq j\leq n,$ then for any t>0,

$$|ta_j-a_{j-1}|\leq(t|a_j|-|a_{j-1}|)\cos\alpha+(t|a_j|+|a_{j-1}|)\sin\alpha.\text{s}$$

Lemma 3 is due to Govil and Rahman [4].

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

F(z) =(1-z)P(z)

$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$

$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_1-a_0)z+a_0$$

$$=-a_nz^{n+1}+a_0+[(k\alpha_n-\alpha_{n-1})-(k-1)\alpha_n]z^n+\sum_{j=2}^{n-1}(\alpha_j-\alpha_{j-1})z^j$$

$$+[(\alpha_1-\tau\alpha_0)+(\tau-1)\alpha_0]z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j .$$

For $|z|\le R$, we have by using the hypothesis

$$|F(z)|\le|a_n|R^{n+1}+|a_0|+R^n[(k-1)|\alpha_n|+(k\alpha_n-\alpha_{n-1})]+\sum_{j=2}^{n-1}|\alpha_j-\alpha_{j-1}|R^j$$

$$+R[|\alpha_1-\tau\alpha_0|+(1-\tau)|\alpha_0|]+\sum_{j=1}^{n}|\beta_j-\beta_{j-1}|R^j .$$

Therefore

$$|F(z)|\le R^{n+1}[|\alpha_n|+|\beta_n|+(k-1)|\alpha_n|+(k\alpha_n-\alpha_{n-1})+\sum_{j=1}^{n-1}(\alpha_j-\alpha_{j-1})$$

$$+|\alpha_0|+|\beta_0|+(\alpha_1-\tau\alpha_0)+(1-\tau)|\alpha_0|+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)]$$

$$=R^{n+1}[k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=1}^{n}|\beta_j|]$$

for $R\ge 1$

and

$$|F(z)|\le\ |a_0|+R[k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

for $R\le 1$.

Hence , by Lemma 2, the number of zeros of F(z) and therefore P(z) in $|z|\le\dfrac{R}{c}$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[R^{n+1}\{k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}[|a_0|+R\{k(|\alpha_n|+\alpha_n)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

for $R\le 1$.

That proves Theorem 1.

Proof of Theorem 2: Consider the polynomial

F(z) =(1-z)P(z)

$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$

$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_1-a_0)z+a_0$$

$$=-a_nz^{n+1}+a_0+[(ka_n-a_{n-1})-(k-1)a_n]z^n+\sum_{j=2}^{n-1}(a_j-a_{j-1})z^j$$

$$+[(a_1-\tau a_0)+(\tau-1)a_0]z .$$

For $|z|\le R$, we have by using the hypothesis and Lemma 3

$$|F(z)|\le|a_n|R^{n+1}+|a_0|+R^n[(k-1)|a_n|+|ka_n-a_{n-1}|]+\sum_{j=2}^{n-1}|a_j-a_{j-1}|R^j$$

$$+[|a_1-\tau a_0|+(1-\tau)|a_0|]R$$

$$\leq |a_n|R^{n+1} + |a_0| + R^n[(k-1)|a_n| + R^n[(k|a_n| - |a_{n-1}|)\cos\alpha$$
$$+ (k|a_n| + |a_{n-1}|)\sin\alpha] + \sum_{j=2}^{n-1}(|a_j| - |a_{j-1}|)\cos\alpha R^j$$
$$+ \sum_{j=2}^{n-1}(|a_j| + |a_{j-1}|)\sin\alpha R^j + [(|a_1| - \tau|a_0|)\cos\alpha + (|a_1| - \tau|a_0|)\sin\alpha$$
$$+ (1-\tau)|a_0|]R$$

which implies

$$|F(z)| \leq R^{n+1}[k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]$$

for $R \geq 1$

and

$$|F(z)| \leq |a_0| + R[k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|]$$

for $R \leq 1$.

Hence , by Lemma 2, it follows that the number of zeros of F(z) and therefore P(z) in $|z| \leq \frac{R}{c}$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0|]}{|a_0|}\ .$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|]}{|a_0|}$$

for $R \leq 1$.

That proves Theorem 3.

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