Research Inventy: International Journal Of Engineering And Science
Issn: 2278-4721, Vol. 2, Issue 5 (February 2013), Pp 01-12
Www.Researchinventy.Com

Mathematical Model For Optimisation Of Modulus Of Rupture Of Concrete Using Osadebe's Regression Theory

[1,]C.E. Okere (Dr.), [2,]D.O. Onwuka(Dr.), [3,]N.N. Osadebe (Prof)
[1,]Civil Engineering, Federal University of Technology, Owerri, Imo State, Nigeria.
[2,]Civil Engineering, Federal University of Technology, Owerri, Nigeria.
[3,] Civil Engineering, University of Nigeria, Nsukka.

Abstract: *The modulus of rupture (MOR) of concrete is a function of the proportions of the constituent materials, namely, cement, water, fine and coarse aggregates. The conventional methods used to determine the mix proportions that will yield a desired modulus of rupture, are laborious, time consuming and expensive. In this paper, a mathematical method based on Osadebe's concrete optimisation theory is formulated for the optimisation of the modulus of rupture of concrete. The model can prescribe all the mixes that will produce a desired modulus of rupture of concrete. It can also predict the modulus of rupture of concrete if the mix proportions are specified .The adequacy of the mathematical model is tested using statistical tools.*

Keywords**:** model; modulus of rupture; Osadebe's theory; optimisation; mix proportions.

I. Introduction

Concrete is a construction material in which strength is very important. The strength is of such utmost importance that it is used as a yardstick for judging other concrete properties such as permeability, durability, fire and abrasion resistances. The strength is usually given in form of compressive strength and flexural strength. The flexural strength is the property of a solid that indicates its ability to resist failure in bending [1] and the modulus of rupture (MOR) of concrete as defined by International Concrete Repair Institute is a measure of the ultimate load bearing capacity of a concrete beam tested in flexure [2].Various methods have been used to study and/or determine the modulus of rupture of concrete [3- 5]. All these methods are based on the conventional approach of selecting arbitrary mix proportions, subjecting concrete samples to laboratory and then adjusting the mix proportions in subsequent tests. Apparently, these methods are time consuming and expensive. In this paper, a mathematical model based on Osadebe's concrete optimisation theory, is formulated for the optimization of the modulus of rupture of concrete.

II. Materials

The materials used in the production of the prototype concrete beams are cement, fine aggregates coarse aggregates and water. Eagle cement brand of Ordinary Portland cement with properties conforming to BS 12 was used in the preparation of the concrete beam specimens [6]. The fine aggregates were fine and medium graded river sand of zone 3 sourced from Otamiri River in Imo State. The coarse aggregates were irregular shaped medium-graded coarse aggregates having a maximum size of 20mm and conforming to BS 882 [7]. They were free from clay lumps and organic materials. Potable water conforming to the specification of EN 1008: was used in the production of the prototype concrete beam specimens [8].

III. Methods

Two methods, namely analytical and experimental methods were used in this work.

3.1 Analytical methods

Here, optimization method is used in formulating a mathematical model for predicting the modulus of rupture of concrete. The model is based on Osadebe's regression theory. A simplex lattice is described as a structural representation of lines joining the atoms of a mixture .The atoms are constituent components of the mixture. For a normal concrete mixture, the constituent elements are water, cement, fine and coarse aggregates and so it gives a simplex of a mixture of four components. Hence the simplex lattice of this four-component mixture is a three-dimensional solid equilateral tetrahedron. Mixture components are subject to the constraint that the sum of all the components must be equal to one [9]. In order words:

$$X_1 + X_2 + X_3 + \ldots\ldots. + X_q = 1 \qquad (1)$$

$$\sum_{i=1}^{q} X_i = 1 \qquad (2)$$

where q is the number of components of a mixture
X_i is the proportion of the ith component in the mixture.

It is impossible to use the normal mix ratios such as 1:2:4 or 1:3:6 at a given water/cement ratio because of the requirement of the simplex that sum of the components must be one. Hence it is necessary to carry out a transformation from actual to pseudo components. The actual components represent the proportion of the ingredients while the pseudo components represent the proportion of the components of the ith component in the mixture i.e. X_1, X_2, X_3, X_4. Considering the four- component mixture tetrahedron simplex lattice, let the vertices of this tetrahedron (principal coordinates) be described by A_1, A_2, A_3, A_4.
The arbitrary mix proportions prescribed for the vertices of the tetrahedron shown in (Figure 1),

A_1 (0.55: 1: 2: 4)
A_2 (0.50: 1: 2.5: 6)
A_3 (0.45: 1: 3: 5.5)
A_4 (0.6: 1: 1.5: 3.5)

are based on past experiences and literature.
Let X represent pseudo components and Z, actual components. For component transformation we use the following equations:

$$X = BZ \qquad (3)$$
$$Z = AX \qquad (4)$$

where A = matrix whose elements are from the arbitrary mix proportions chosen when 'equation (4)' is opened and solved mathematically.
B = the inverse of matrix A.
X = matrix of pseudo components. This is obtained from (Figure 2).
Expanding and using 'equations (3) and (4)', the actual components Z were determined and presented in (Table 1) [9].

3.2 Formulation of the optimisation model

Osadebe's regression model is used in the formulation of the mathematical model for the optimization of the modulus of rupture of concrete. Osadebe assumed that the response function, F(z) given by 'equation (1)' is continuous and differentiable with respect to its predictors, Zi [10] .

$$F(z) = F(z^{(0)}) + \sum[\partial F(z^{(0)}) /\partial z_i](z_i - z_i^{(0)}) + ½! \sum\sum[\partial^2 F(z^{(0)}) / \partial z_i \partial z_j](z_i - z_i^{(0)}) (z_j - z_j^{(0)}) + ½! / \sum\sum[\partial^2 F(z^{(0)}) / \partial z_i^2](z_i - z_i^{(0)})^2 + \ldots\ldots \qquad (5)$$

where $1\leq i\leq 4$, $1\leq i\leq 4$, $1\leq j\leq 4$,and $1\leq i\leq 4$ respectively.
By making use of Taylor's series, the response function could be expanded in the neighbourhood of a chosen point:

$$Z^{(0)} = Z_1^{(0)}, Z_2^{(0)}, Z_3^{(0)}, Z_4^{(0)}, Z_5^{(0)} \qquad (6)$$

Without loss of generality of the formulation, the point $z^{(0)}$ will be chosen as the origin for convenience sake. It is worthy of note here that the predictor, z_i is not the actual portion of the mixture component rather it is the ratio of the actual portions to the quantity of concrete. For convenience sake, let z_i be the fractional portion and s_i be the actual portions of the mixture components.
If the total quantity of concrete is designated s, then

$$\sum s_i = s \qquad (7)$$

For concrete of four components, $1\leq i\leq 4$ and so 'equation (7)' becomes:

$$s_1 + s_2 + s_3 + s_4 = s \qquad (8)$$

If the total quantity of concrete required is a unit quantity, then 'equation (8) should be divided throughout by s. Hence

$$s_1/s + s_2/s + s_3/s + s_4/s = s/s \qquad (9)$$

But fractional portions, $z_i = s_i/s$ (10)
Substituting 'equation (10)' into 'equation (9)' gives 'equation (11)'

$$z_1 + z_2 + z_3 + z_4 = 1 \qquad (11)$$

Experience has shown that the coefficients of regression obtained when $\sum z = 1$ are mostly too large that the regression becomes too sensitive. As a result when the values of predictors outside the ones used in formulating the model are used to predict the response, the regression gives outrageous values. To correct this shortcoming, a system of z that will make $\sum z = 100$ will be adopted. Thus multiplying 'equation (11)' by 100 yields:

$$100z_1 + 100z_2 + 100z_3 + 100z_4 = 100 \quad (12)$$

Let $100z_i = Z$ (13)

Therefore, 'equation (12)' becomes:

$$Z_1 + Z_2 + Z_3 + Z_4 = 10 \quad (14)$$

In the formulation of the regression equation, the point, $z^{(0)}$ was chosen as the origin.
This implies that $z^{(0)} = 0$ and so

$$z_1^{(0)} = 0,\ z_2^{(0)} = 0,\ z_3^{(0)} = 0 \text{ and } z_4^{(0)} = 0$$

Let

$$b_0 = F(0) \quad (15)$$

$$b_i = \partial F(0) / \partial z_i \quad (16)$$

$$b_{ij} = \partial^2 F(0) / \partial z_i \partial z_j \quad (17)$$

$$bii = \partial^2 F(0) / \partial z_i^2 \quad (18)$$

Substituting 'equations (15) – (18)' into 'equation (5)' gives:

$$F(z) = b_0 + \sum b_i z_i + \sum\sum b_{ij} z_i z_j + \sum b_{ii} z_i^2 + \ \ldots.. \quad (19)$$

where $1 \leq i \leq 4$ and $1 \leq j \leq 4$

Multiplying 'equation (11)' by b_0 gives the expression for b_0 i.e. 'equation (20)'

$$b_0 = b_0 z_1 + b_0 z_2 + b_0 z_3 + b_0 z_4 \quad (20)$$

Multiplying 'equation (11)' by z_1, z_2, z_3 and z_4, and rearranging the products, gives 'equation (21)-(24)'

$$z_1^2 = z_1 - z_1z_2 - z_1z_3 - z_1z_4 \quad (21)$$

$$z_2^2 = z_2 - z_1z_2 - z_2z_3 - z_2z_4 \quad (22)$$

$$z_3^2 = z_3 - z_1z_3 - z_2z_3 - z_3z_4 \quad (23)$$

$$z_4^2 = z_4 - z_1z_4 - z_2z_4 - z_3z_4 \quad (24)$$

Substituting 'equations (20) – (24)' into equation (19)' and simplifying yields 'equation (25)'

$$Y = \alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3 + \alpha_4z_4 + \alpha_{12} z_1z_2 + \alpha_{13}z_1z_3 + \alpha_{14}z_1z_4 + \alpha_{23}z_2z_3 + \alpha_{24}z_2z_4 + \alpha_{34}z_3z_4 \quad (25)$$

where

$$\alpha_i = b_0 + b_i + b_{ii} \quad (26)$$

and

$$\alpha_{ij} = b_{ij} - b_{ii} - b_{jj} \quad (27)$$

In general, 'equation (25)' is given as

$$Y = \sum \alpha_i z_i + \sum \alpha_{ij} z_i z_j \quad (28)$$

where $1 \leq i \leq j \leq 4$

'Equations (25) and (28)' are the optimization model equations when the system of $\Sigma z = 1$ is used. But the system of $\Sigma Z = 100$ is adopted here and 'equation (28)' becomes:

$$Y = \sum \alpha_i Z_i + \sum \alpha_{ij} Z_i Z_j \quad (29)$$

where $1 \leq i \leq j \leq 4$

Y is the response function at any point of observation, z_i and Z_i are the predictors and α_i and α_i are the coefficients of the optimization model equations when systems $\sum z_i = 1$ and $\sum Z_i = 100$ are used respectively.

3.3 Determination of the coefficients of the optimization model equation

Different points of observation will have different responses with different predictors at constant coefficients. At nth observation point, $Y^{(n)}$ will correspond with $Z_i^{(n)}$. That is,

$$Y^{(n)} = \sum \alpha_i Z_i^{(n)} + \sum \alpha_{ij} Z_i^{(n)} Z_j^{(n)} \quad (30)$$

where $1 \leq i \leq j \leq 4$ and n = 1,2,3, …………. 10

'Equation (30)' can be put in matrix from as

$$[Y^{(n)}] = [Z^{(n)}] \{\alpha\} \quad (31)$$

Rearranging 'equation (31)' gives:

$$\{\alpha\} = [Z^{(n)}]^{-1} [Y^{(n)}] \quad (32)$$

The actual mix proportions, $s_i^{(n)}$) and the corresponding fractional portions, $z_i^{(n)}$ are presented in (Table 2). These values of the fractional portions $Z^{(n)}$ were used to develop $Z^{(n)}$ matrix and the inverse of $Z^{(n)}$ matrix presented in (Table 3). The values of $Y^{(n)}$ matrix are determined from laboratory tests and presented in (Table 4). With the values of the matrices $Y^{(n)}$ and $Z^{(n)}$ known, it is easy to determine the values of the constant coefficients of 'equation (31)'.

3.4 Experimental Method

The actual components as transformed from 'equation (4)' and (Table 1) were used to measure out the quantities water (Z_1), cement (Z_2), sand (Z_3), and coarse aggregates (Z_4) in their respective ratios for the modulus of rupture of concrete test. For instance, the actual ratio for the test number 20 means that the concrete mix ratio is 1: 2.5: 5.3 at 0.5 free water/cement ratio. A total of 25 mix ratios were used to produce 50 prototype concrete beams measuring 150mm ^{x} 150mm ^{x} 600mm that were cured and tested on the 28th day. Fifteen out of 20 mix ratios were used as control mix ratios to produce 30 beams for the confirmation of the adequacy of the mixture design model given by 'equation (25)'. The beams were then tested for flexural strength (i.e. modulus of rupture) using the hand operated flexural testing machine. The symmetrical two point loading system was used. The load under which the beam specimen failed was recorded and used to compute the modulus of rupture of the prototype beams [11].

IV. Results And Analysis

The test result of the modulus of rupture of concrete (Y_i) based on day 28-day strength, is presented as part of (Table 4).
The flexural strength (modulus of rupture) was obtained from the following equation:

$$\sigma = WL/\ bh^2 \qquad (33)$$

where σ is the modulus of rupture in Mega Pascals (MPa) or Newtons per millimeters squared (Nmm^{-2}).
W = maximum load in Newtons (N).
L = the distance between supporting rollers in millimetres (mm).
b and h are the lateral dimensions of the specimen, in millimetres.
The values of the mean of responses, Y and the variances of replicates S_i^2 presented in columns 5 and 8 of (Table 4) are gotten from the following 'equations (34) and (35)':

$$Y = \sum_{i=1}^{n} Y_i/n \qquad (34)$$

$$S^2_i = [1/(n-1)]\{\sum Y_i^2 - [1/n(\sum Y_i)^2]\} \qquad (35)$$

where $1 \leq i \leq n$ and this equation is an expanded form of 'equation(36)'

$$S^2_i = [1/(n-1)][\sum_{i=1}^{n} (Y_i - Y)^2] \qquad (36)$$

where Y_i = responses
Y = mean of the responses for each control point
n = number of parallel observations at every point
n−1 = degree of freedom
S^2_i = variance at each design point
Considering all the design points, the number of degrees of freedom, V_e is given as

$$V_e = \sum N-1 \qquad (37)$$
$$= 25 - 1$$
$$= 24$$

where N is the number of points

Replication variance, $S^2_y = (1/Ve) \sum_{i=1}^{N} S_i^2$ (38)

$=22.172/24 = 0.924$

where S_i^2 is the variance at each point
Using 'equations (37) and (38)', the replication error, S_y can be determined as follows:

$$S_y = \sqrt{S^2_y} \qquad (39)$$
$$= 0.961$$

This replication error value was used below to determine the t-statistics values for Scheffe's simplex model.

4.1 Determination of the optimisation model based on Osadebe's theory

Substituting the values of $Y^{(n)}$ from test results presented in (Table 4) into 'equation (32)' gives the following values of the coefficients of the model developed i.e. 'equation (25)'.

$\alpha_1 = 448366.053$ $\alpha_2 = - 3.70408.3989$ $\alpha_3 = - 13065.39191$
$\alpha_4 = 3.283969106$ $\alpha_5 = 796.3475619$ $\alpha_6 = -5211.73858$
$\alpha_7 = 5857.977279$ $\alpha_8 = 5309.170464$ $\alpha_9 = 4734.703595$
$\alpha_{10} = - 6.916241183$

Substituting the values of these coefficients into ‘equation (31)’ yields:

$$Y = 448366.053Z_1 - 370408.3989Z_2 - 13065.39191Z_3 + 3.283969106Z_4 + 796.3475619Z_1Z_2 - 5211.73858Z_1Z_3 - 5857.977279Z_1Z_4 + 5309.170464Z_2Z_3 + 4734.703595Z_2Z_4 - 6.916241183Z_3Z_4 \quad (40)$$

‘Equation (40)’ is Osadebe’s mathematical model of modulus of rupture of concrete based on the 28-day strength.

4.2 Test of the adequacy of the model

Osadebe’s model equation was tested for adequacy against the controlled experimental results. It will be recalled that the hypothesis for this mathematical model are as follows: Null Hypothesis (H_o): There is no significant difference between the experimental and the theoretically expected results at an α- level of 0.05. Alternative Hypothesis (H_1): There is a significant difference between the experimental and theoretically expected results at an α –level of 0.05. The student’s t-test and fisher test statistics were used for this test. The expected values ($Y_{\text{predicted}}$) for the test control points were obtained by substituting the values of Z_i from Z^n matrix into the model equation i.e. ‘equation (40)’. These values were compared with the experimental result (Y_{observed}) from (Table 5).

4.3 Student’s test

For this test, the parameters Δ_Y, ϵ and t are evaluated using the following equations respectively

$$\Delta_Y = Y_{(\text{observed})} - Y_{(\text{predicted})} \quad (41)$$

$$\epsilon = (\sum a_i^2 + \sum a_{ij}^2) \quad (42)$$

$$t = \Delta y\sqrt{n} / (Sy\sqrt{1+\epsilon}) \quad (43)$$

where ϵ is the estimated standard deviation or error,
t is the t-statistics,
n is the number of parallel observations at every point
S_y is the replication error
a_i and a_{ij} are coefficients while i and j are pure components
$a_i = X_i(2X_i-1)$
$a_{ij} = 4X_iX_j$
$Y_{obs} = Y_{(observed)}$ = Experimental results
$Y_{pre} = Y_{(predicted)}$ = Predicted results
At significant level, α = 0.05, $t_{\alpha/1}(V_e) = t_{0.05/10} = t_{0.005(14)} = 2.977$. The t-value is obtained from standard t-statistics table.
Since this is greater than any of the t- values calculated in (Table 5), we accept the Null hypothesis. Hence the model is adequate.

4.4 Fisher Test

For this test, the parameter y, is evaluated using the following equation:

$$y = \sum Y/n \quad (44)$$

where Y is the response and n the number of responses.
Using variance, $S^2 = [1/(n-1)][\sum (Y-y)^2]$ and $y = \sum Y/n$ for $1 \le i \le n$ (45)
Therefore from (Table 6),
$S^2_{(obs)} = 10.64933/14 = 0.761$ and $S^2_{(pre)} = 12.36237/14 = 0.883$
But the fisher test statistics is given by:

$$F = S^2_1 / S^2_2 \quad (46)$$

where S^2_1 is the larger variance
Hence $S^2_1 = 0.833$ and $S^2_2 = 0.761$
Therefore, F = 0.833/0.761 = 1.095
From standard Fisher Table, $F_{0.95}(14,14) = 2.41$.Hence the regression equation is adequate.

4.5 Comparison of results

The results obtained from the model were compared with those obtained from the experiment, as presented in (Table 7) A comparison of the predicted results with the experimental results shows that the percentage difference ranges from a minimum of 2.95% to a maximum of 7.07%, which is insignificant.

V. Conclusion

1. Osadebe’s regression model using Taylor’s series has been applied and used successfully to develop mathematical models for optimization of modulus of rupture of concrete.

2. The modulus of rupture of concrete is a function of the proportions of the ingredients (cement, water, sand and coarse aggregate) of the concrete.
3. The student's t-test and the fisher test used in the statistical hypothesis showed that the model developed is adequate.
4. Since the maximum percentage difference between the experimental result and the predicted result is insignificant (i.e.7.07%), the optimisation model will yield accurate values of modulus of rupture if given the mix proportions and vice versa.

Vi Nomenclature

∑ summation
≤ less or equal to
– subtraction
+ addition
x multiplication
√ square root
α coefficient
b constant coefficient
σ modulus of rupture of concrete
mm millimetres
Δ_Y change in Y
[bracket
(parenthesis
X_i proportion of the ith component in the mixture.
q number of components of a mixture
A vertice of tetrahedron
X pseudo components (variables)
Z actual components (variables)
B inverse of matrix A
F(z) response function
z_i predictor(fractional portion)
s_i actual portions
α_i coefficients of optimisation model equation
Yi response(modulus of rupture of concrete)
Y mean of responses
n number of parallel observations at every point
N number of points
S^2_1 variance at each design point
ϵ estimated standard deviation or error
t t-statistics
S_y replication error
a_i and a_{ij} coefficients
i and j pure components
$Y_{obs} = Y_{(observed)}$ = Experimental results
$Y_{pre} = Y_{(predicted)}$ = Predicted results
F Fisher test statistics

References

[1] Neville, A.M. (1996). Properties of concrete. England: Longman.
[2] International Concrete Repair Institute Concrete Repair Terminology [online]. Available from: http://www.google.com/search.
[3] Hueste, M.B.D.; Chompreda, P.; Trejo David; Clinic Daren B.H.; and Feating, P.B.(2004). Mechanical properties of high strength concrete for prestressed members. ACI Structural Journal.
[4] Kohler, E.; Ali, A.; and Harvey, J. (2005). Flexural fatigue life of hydraulic cement concrete beam. Department of Transportation, Partnered Pavement Research Centre, University of Carlifornia, Davies and Berkeley.
[5] Elinwa, A.U.; and Ejeh, S.P. (2005). Characteristics of sisal fibre reinforced concrete. Journal of Civil Engineering Research and Practice, 2 (1), 1-14.
[6] British Standards Institution, 1978. BS 12: 1978. Specification for Portland cement.

[7] British Standards Institution (1992). BS 882: 1992. Specification for aggregates from natural sources for concrete.

[8] British Standards Institution (2002). BS EN 1008: 2002. Mixing water for concrete -Specification for sampling, testing and assessing the suitability of water, including water recovered from processes in the concrete industry, as mixing water for concrete.

[9] Scheffe, H. (1958). Experiments with Mixtures. Royal Statistical Society Journal, 20 (B). 344-360.

[10] Osadebe, N.N. (2003). Generalised mathematical modeling of compressive strength of normal concrete as a multi-variant function of the properties of its constituent components. A paper delivered at the college of Engineering, University of Nigeria, Nsukka.

[11] Okere, C.E. (2006). Mathematical Models for Optimisation of Modulus of Rupture of Concrete. Thesis (M. Eng). Federal University of Technology Owerri.

Table 1. Actual Components Z (points 1 to 25)

N	X_1	X_2	X_3	X_4	Response	Z_1	Z_2	Z_3	Z_4
1	1	0	0	0	Y_1	0.55	1	2	4
2	0	1	0	0	Y_2	0.50	1	2.5	6
3	0	0	1	0	Y_3	0.45	1	3	5.5
4	0	0	0	1	Y_4	0.6	1	1.5	3.5
5	0.5	0.5	0	0	Y_{12}	0.525	1	2.25	5
6	0.5	0	0.5	0	Y_{13}	0.5	1	2.5	4.75
7	0.5	0	0	0.5	Y_{14}	0.575	1	1.75	3.75
8	0	0.5	0.5	0	Y_{23}	0.475	1	2.75	5.75
9	0	0.5	0	0.5	Y_{24}	0.55	1	2	4.75
10	0	0	0.5	0.5	Y_{34}	0.525	1	2.25	4.5
Control points within the factor space									
11	0.5	0.25	0.25	0	C_1	0.5125	1	2.375	4.875
12	0.25	0.25	0.25	0.25	C_2	0.525	1	2.25	4.75
13	0	0.25	0.25	0.5	C_3	0.5375	1	2.125	4.625
14	0	0.25	0	0.75	C_4	0.575	1	1.75	4.125
15	0.75	0	0.25	0	C_5	0.525	1	2.25	4.375
16	0	0.5	0.25	0.25	C_6	0.5125	1	2.375	5.25
17	0.25	0	0.5	0.25	C_7	0.5125	1	2.375	4.625
18	0.75	0.25	0	0	C_8	0.5375	1	2.125	4.5
19	0	0.75	0.25	0	C_9	0.4875	1	2.625	5.875
20	0	0.4	0.4	0.2	C_{10}	0.5	1	2.5	5.3
Control points outside the factor space									
21	0.5	0.5	0.5	0.5	C_{11}	1.05	2	4.5	9.5
22	0.25	0	0.25	0	C_{12}	0.35	0.5	1.375	2.875
23	0.5	0	0.5	0.5	C_{13}	0.8	1.5	3.25	6.5
24	0.25	0.25	0.25	0	C_{14}	0.375	0.75	1.875	3.875
25	0	0.5	0.5	0.25	C_{15}	0.625	1.25	3.125	6.625

Table 2. Values of actual mix proportions and the corresponding fractional portions

N	S_1	S_2	S_3	S_4	RESPONSE	Z_1	Z_2	Z_3	Z_4
1	0.55	1	2	4	Y_1	7.285	13.245	26.490	52.980
2	0.5	1	2.5	6	Y_2	5.000	10.000	25.000	60.000
3	0.45	1	3	5.5	Y_3	4.523	10.050	30.151	55.276
4	0.6	1	1.5	3.5	Y_4	9.091	15.152	22.727	53.030
5	0.525	1	2.25	5	Y_{12}	5.983	11.396	25.641	56.980
6	0.5	1	2.5	4.75	Y_{13}	5.714	11.429	28.571	54.286
7	0.575	1	1.75	3.75	Y_{14}	8.127	14.134	24.735	53.004
8	0.475	1	2.75	5.75	Y_{23}	4.762	10.025	27.569	57.644
9	0.55	1	2	4.75	Y_{24}	6.627	12.048	24.096	57.229
10	0.525	1	2.25	4.5	Y_{34}	6.344	12.085	27.190	54.381

Table 3. Z $^{(n)}$ matrix and inverse of $Z^{(n)}$ matrix

$Z^{(n)}$ matrix

Z_1	Z_2	Z_3	Z_4	Z_1Z_2	Z_1Z_3	Z_1Z_4	Z_2Z_3	Z_2Z_4	Z_3Z_4
7.2848	13.2450	26.4901	52.9801	96.4872	192.9751	385.9494	350.8614	701.7214	1403.44
5.0000	10.0000	25.000	60.0000	50.0000	125.0000	300.0000	250.0000	600.0000	1500.00
4.5226	10.0503	30.1508	55.2764	45.4535	136.3600	249.9930	303.0246	555.5444	1666.62
9.0909	15.1515	22.7273	53.0303	137.7408	206.6116	482.0932	344.3527	803.4886	1205.23
5.9829	11.3960	25.6410	56.9801	68.1811	153.4075	340.9062	292.2048	649.3452	1461.02
5.7143	11.4286	28.5714	54.2857	65.3065	163.2656	310.2048	326.5311	620.4096	1551.01
8.1272	14.1343	24.7350	53.0035	114.8723	201.0263	430.7700	349.6119	749.1674	1311.04
4.7619	10.0251	27.5689	57.6441	47.7385	131.2803	274.4954	276.3810	577.8879	1589.18
6.6265	12.0482	24.0964	57.2289	79.8374	159.6748	379.2273	290.3182	689.5052	1379.01
6.3444	12.0846	27.1903	54.3807	76.6695	172.5061	345.0129	328.5839	657.1690	1478.62

Inverse of $Z^{(n)}$ matrix

5923.0126	-7256.4049	-98031.5654	163046.0717	-918988.7094	-392203.8683	-534341.5107	350976.4150	383909.9865	856966.6809
1896.1943	42755.8993	-76497.8664	82688.8657	120016.6747	164982.0897	-70696.6484	-73920.8754	-134553.9780	67122.1197
4093.7500	-5705.2756	-1845.6822	-8836.3837	584.1348	16161.2454	16611.5960	7780.1215	8783.7940	-37627.2883
0.5183	1.2730	-0.2676	0.5967	-5.1461	-0.7308	-1.8511	0.2960	1.7086	3.6132
-584.4199	-870.5949	3500.4840	-4782.6618	13281.8725	3014.0260	11000.5758	-4417.2306	-3578.9226	-16563.1319
1394.9796	-448.4334	716.2966	-2263.1545	8325.4328	4984.4571	6305.2502	-2458.7385	-2665.4755	-11100.6560
2641.2865	482.9282	957.6803	-1092.5493	10935.2216	4189.5831	4851.5227	-4619.0266	-5272.7973	-7791.2772
1033.6385	0.6347	1023.7686	-185.7534	-1411.4036	-3243.6375	-611.5452	176.5671	778.9482	2438.7816
1730.4842	-735.3958	781.9636	-1230.3211	-2509.0308	-1850.4568	1075.8870	1571.1467	2420.8098	-1255.0877
-109.1658	98.0425	16.2059	142.1187	168.9416	-134.8258	-215.1951	-188.7414	-231.3071	453.9265

Table 4. Test Results and Replication Variance

EXP NO	Replicates	Response Y_i (N/mm^2)	Response Symbol	Y	$\sum Y_i$	$\sum Yi^2$	Si^2
1	1A IB	5.96 6.44	Y_1	6.2	12.40	77.00	0.120
2	2B 2B	5.82 5.86	Y_2	5.8	11.68	68.21	0.000
3	3A 3B	4.84 6.33	Y_3	5.6	11.17	63.49	1.105
4	4A 4B	5.73 6.04	Y_4	5.9	11.77	69.31	0.043
5	5A 5B	4.58 5.72	Y_{12}	5.2	10.30	53.69	0.645
6	6A 6B	5..16 3.70	Y_{13}	4.4	8.86	40.32	1.070
7	7A 7B	6.32 7.38	Y_{14}	6.9	13.70	94.41	0.565
8	8A 8B	4.49 4.46	Y_{23}	4.5	8.95	40.05	0.000
9	9A 9B	5.16 7.38	Y_{24}	6.3	12.54	81.09	2.464
10	10A 10B	6.63 5.11	Y_{34}	5.9	11.74	70.07	1.156
11	11A 11B	4.58 5.07	C_1	4.8	9.65	46.68	0.119
12	12A 12B	4.09 6.80	C_2	5.4	10.89	62.97	3.674
13	13A 13B	5.87 6.14	C_3	6.0	12.01	72.16	0.040
14	14A 14B	6.54 5.68	C_4	6.1	12.22	75.03	0.366
15	15A 15B	4.72 5.68	C_5	5.2	10.40	54.54	0.460
16	16A 16B	4.80 3.64	C_6	4.2	8.44	36.29	0.673
17	17A 17B	3.96 5.07	C_7	4.5	9.03	41.39	0.619
18	18A 18B	5.29 3.16	C_8	4.2	8.45	37.97	2.269
19	19A 19B	4.49 3.15	C_9	3.8	7.64	30.08	0.895
20	20A 20B	5.13 5.48	C_{10}	5.3	10.61	56.35	0.064
						Σ	16.347
CONTROL OUTSIDE FACTOR SPACE							
21	21A 21B	4.09 6.80	C_{11}	5.4	10.89	62.97	3.674
22	22A 22B	5.24 5.82	C_{12}	5.5	11.06	61.33	0.168
23	23A 23B	5.51 5.69	C_{13}	5.6	11.20	62.74	0.020
24	24A 24B	3.02 5.02	C_{14}	4.0	8.04	34.32	1.959
25	25A 25B	3.20 3.07	C_{15}	3.1	6.27	19.66	0.004
						ΣΣ	22.172

Table 5. T –Statistics for test control points

N	CN	i	j	a_i	a_{ij}	a^2_i	a^2_{ij}	ϵ	Y_{obs}	Y_{pre}	Δ_Y	t
		1	2	0	0.5	0	0.25					
		1	3	0	0.5	0	0.25					
		1	4	0	0	0	0					
1	C_1	2	3	-0.125	0.25	0.0156	0.0625		4.8	4.4	0.4	0.46
		2	4	-0.125	0	0.0156	0					
		3	4	-0.125	0	0.0156	0					
		4	-	0	-	0	0					
					Σ	0.0468	0.5625	0.6093				
		1	2	-0.125	0.25	0.0156	0.0625					
		1	3	-0.125	0.25	0.0156	0.0625					
		1	4	-0.125	0.25	0.0156	0.0625					
2	C_2	2	3	-0.125	0.25	0.0156	0.0625		5.4	5.02	0.38	0.46
		2	4	-0.125	0.25	0.0156	0.0625					
		3	4	-0.125	0.25	0.0156	0.0625					
		4	-	-0.125	-	0.0156	-					
					Σ	0.1092	0.375	0.4842				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
3	C_3	2	3	-0.125	0.25	0.0156	0.0625		6.0	5.22	0.78	0.90
		2	4	-0.125	0.5	0.0156	0.25					
		3	4	-0.125	0.5	0.0156	0.25					
		4	-	0	-	0	-					
					Σ	0.0468	0.5625	0.6093				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
4	C_4	2	3	-0.125	0	0.0156	0		6.1	6.28	0.18	0.20
		2	4	-0.125	0.75	0.0156	0.5625					
		3	4	0	0	0	0					
		4	-	0.375	-	0.1406	-					
					Σ	0.1718	0.5625	0.7343				
		1	2	0.375	0	0.1406	0					
		1	3	0.375	0.75	0.1406	0.5625					
		1	4	0.375	0	0.1406	0					
5	C_5	2	3	0	0	0	0		5.2	5.57	0.55	1.36
		2	4	0	0	0	0					
		3	4	-0.125	0	0.0156	0					
		4	-	0	-	0	-					
					Σ	0.4374	0.5625	0.9999				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
6	C_6	2	3	0	0	0	0.25		4.2	4.4	-0.2	0.23
		2	4	0	0.5	0	0.25					
		3	4	-0.125	0.25	0.0156	0.0625					
		4	-	-0.125	-	0.0156	-					
					Σ	0.0312	0.5625	0.5937				
		1	2	-0.125	0	0.0156	0					
		1	3	-0.125	0.5	0.0156	0.25					
		1	4	-0.125	0.25	0.0156	0.0625					
7	C_7	2	3	0	0	0	0		4.5	5.4	0.19	0.22
		2	4	0	0	0	0					
		3	4	0	0.5	0	0.25					
		4	-	-0.125	-	0.0156	-					
					Σ	0.0624	0.5625	0.6249				
		1	2	0.375	0.75	0.1406	0.5625					
		1	3	0.375	0	0.1406	0					
		1	4	0.375	0	0.1406	0					
8	C_8	2	3	-0.125	0	0.0156	0		4.2	5.63	1.43	1.48
		2	4	-0.125	0	0.0156	0					
		3	4	0	0	0	0					
		4	-	0	0	0	0					
					Σ	0.453	0.5625	1.0155				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
9	C_9	2	3	0.375	0.75	0.1406	0.5625		3.8	3.41	0.39	0.42
		2	4	0.375	0	0.1406	0					
		3	4	-0.125	0	0.0156	0					
		4	-	0	-	0	-					

					Σ	0.2968	0.5625	0.8593				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
10	C_{10}	2	3	-0.08	0.64	0.0064	0.4096		5.3	3.54	1.76	2.02
		2	4	-0.08	0.32	0.0064	0.1024					
		3	4	-0.08	0.32	0.0064	0.1024					
		4	-	-0.12	-	0.0144	-					
					Σ	0.0336	0.6144	0.648				
		1	2	0	1	0	1					
		1	3	0	1	0	1					
		1	4	0	1	0	1					
11	C_{11}	2	3	0	1	0	1		5.4	5.02	0.38	0.20
		2	4	0	1	0	1					
		3	4	0	1	0	1					
		4	-	0	-	0	1					
					Σ	0	7	7.0				
		1	2	-0.125	0	0.0156	0					
		1	3	-0.125	0.25	0.0156	0.0625					
		1	4	-0.125	0	0.0156	0					
12	C_{12}	2	3	0	0	0	0					
		2	4	0	0	0	0		5.5	4.26	1.24	1.72
		3	4	-0.125	0	0.0156	0					
		4	-	0	-	0	0					
					Σ	0.0624	0.0625	0.1249				
		1	2	0	0	0	0					
		1	3	0	1	0	1					
		1	4	0	1	0	1					
13	C_{13}	2	3	0	0	0	0		5.6	5.82	0.22	0.16
		2	4	0	0	0	0					
		3	4	0	1	0	1					
		4	-	0	-	0	-					
					Σ	0	3	3				
		1	2	-0.125	0.25	0.0156	0.0625					
		1	3	-0.125	0.25	0.0156	0.0625					
		1	4	-0.125	0	0.0156	0					
14	C_{14}	2	3	-0.125	0.25	0.0156	0.0625		4.0	3.5	0.5	1.09
		2	4	-0.125	0	0.0156	0					
		3	4	-0.125	0	0.0156	0					
		4	-	0	-	0.0156	-					
					Σ	0.1875	0.1875	0.2811				
		1	2	0	0	0	0					
		1	3	0	0	0	0					
		1	4	0	0	0	0					
15	C_{15}	2	3	0	1	0	1		3.1	3.54	0.44	0.41
		2	4	0	0.5	0	0.25					
		3	4	0	0.5	0	0.25					
		4	-	-0.125	-	0.0156	-					
					Σ	0.0156	1.5	1.5156				

Table 6: F-Statistics for the controlled points

Response Symbol	$Y_{(observed)}$	$Y_{(predicted)}$	$Y_{(obs)}-y_{(obs)}$	$Y_{(pre)}-y_{(pre)}$	$Y_{(obs)}-y_{(obs)}^2$	$(Y_{(pre)}-y_{(pre)})^2$
C_1	4.80	4.40	-0.07333	-0.29867	0.005378	0.089202
C_2	5.40	5.02	0.526667	0.321333	0.277378	0.103255
C_3	6.00	5.22	1.126667	0.521333	1.269378	0.271788
C_4	6.10	6.28	1.226667	1.581333	1.504711	2.500615
C_5	5.20	5.75	0.326667	1.05133	0.106711	1.105302
C_6	4.20	4.40	-0.67333	0.29867	0.453378	0.089202
C_7	4.50	4.69	-0.37333	-0.00867	0.139378	7.51E-05
C_8	4.20	5.63	-0.67333	0.931333	0.453378	0.867382
C_9	3.80	3.41	-1.07333	-1.28867	0.152044	1.660662
C_{10}	5.30	3.54	0.426667	-1.15867	0.182044	1.342508
C_{11}	5.40	5.02	0.526667	0.321333	0.277378	0.103255
C_{12}	5.50	4.26	0.626667	-0.43867	0.392711	0.192428
C_{13}	5.60	5.82	0.726667	1.121333	0.528044	1.257388
C_{14}	4.00	3.50	-0.87333	-1.19867	0.762711	1.436802
C_{15}	3.10	3.54	-1.77333	-1.15867	3.144711	1.342508
Sum	73.10	70.48			10.64933	12.36237
Mean	$y_{(obs)}=4.87$	$y_{(pre)}=4.70$				

Table 7: Comparison of some Predicted Result with Experimental Results

S/N	Experimental Result (N/mm^2)	Predicted Result (N/mm^2)	Percentage Difference
1	4.50	4.69	4.22
2	5.40	5.02	7.07
3	5.60	5.82	3.93
4	6.10	6.28	2.95
5	4.50	4.69	4.22
6	4.20	4.40	4.76

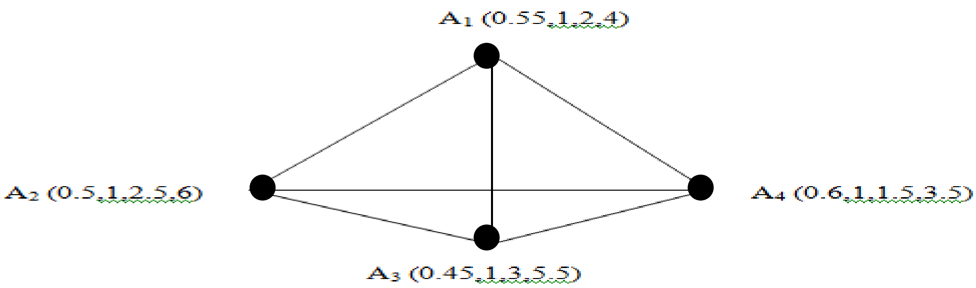

Fig 1: Vertices of a (4,2) lattice (actual)

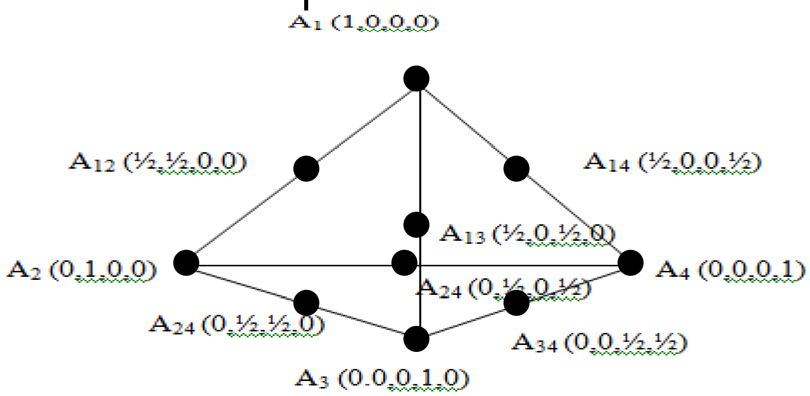

Fig 2: Vertices of a (4,2) lattice (pseudo)