RESEARCH INVENTY: International Journal of Engineering and Science
ISBN: 2319-6483, ISSN: 2278-4721, Vol. 2, Issue 1
(January 2013), PP 65-69
www.researchinventy.com

Fixed Point Theorems for Eight Mappings on Menger Space through Compatibility

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***Abstract**: In this paper we prove a common fixed point theorem for eight mappings on Menger spaces using the notion of compatibility and continuity of maps.*

***Key words:** Menger Space, Compatible Mappings, Weak-compatible Mappings, Common fixed point.*

I. Introduction:

There have been a number of generalization of metric space.One such generalization is Menger space initiated by Menger[1].it is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$.Schweizer and sklar[3] studied this concept and gave some fundamental result on this space.Sehgal,Bharucha-Reid[5] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed point theory in Menger space. In 1991 Mishra [2] introduced the concept of compatible maps in Menger spaces and gave some common fixed point theorems.In 1980,singh and singh [9] gave the following theorem on metric space for self maps which is use to our main result:

Theorem (A): Let P, Q and T be self maps of a metric space (X, d) such that

(1) PT=TP and QT=TQ,
(2) $P(X) \cup Q(X) \subseteq T(X)$
(3) T is continuous ,and
(4) $d(Px, Qy) \leq c\lambda(x, y)$,where
(5) for all $x,y\in X$ and $0\leq c<1$.Futher if (5) X is complete then P, Q and T have a unique common fixed point $z\in X$.

In 2006, Bijendra Singh and Shishir Jain [8] introduced fixed point theorems in Menger space through semi-compatibility and gave the following fixed point theorem for six mappings:

Theorem (B): Let A, B, S, T, L and M are self maps on a complete Menger space (X, F, min) satisfying:

(a) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$.
(b) AB = BA, ST = TS, LB = BL, MT = TM.
(c) Either AB or L is continuous.
(d) (L, AB) is semi-compatible and (M, ST) is weak-compatible.
(e) There exists $K \in (0, 1)$ such that,
$F_{Lp, Mq}(kx) \geq \min\{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, Mq((2-\beta)x)}, F_{ABp, STq}(x)\}$.

In this paper we generalize and extend the result of Bijendra Singh and Shishir Jain [8] for eight mapping opposed to six mappings in complete Menger space using the concept of compatibility.

For the sake of convenience we give some definitions.

Definition (1.1): A probabilistic metric space (PM-space) is a ordered pair (X, F) where X is an abstract act of elements and $F: X \times X \to L$, defined by $(p, q) \to F_{p, q}$ where L is the set of all distribution functions i.e. $L = \{F_{p, q} / p, q \in X\}$,If the function $F_{p, q}$ satisfy:

(a) $F_{p, q}(x) = 1$ for all $x > 0$, if and only if p = q,
(b) $F_{p, q}(0) = 0$,
(c) $F_{p, q} = F_{q, p}$,
(d) If $F_{p, q}(x) = 1$ and $F_{q, r}(y) = 1$ then $F_{p, r}(x + y) = 1$.

Definition (1.2): A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if,

(a) $t(a, 1) = a$, $t(0, 0) = 0$,
(b) $t(a, b) = t(b, a)$ (symmetry property),
(c) $f(c, d) \geq t(a, b)$ for $c \geq a$, $d \geq b$,
(d) $t(t(a, b), c) = t(a, t(b, c))$.

Definition (1.3): A Menger space is a triplet (X, F, t) where (X, F) is a PM-space and t is a t-norm such that for all $p, q, r \in X$ and for all $x, y \geq 0$, $F_{p, r}(x+y) \geq t(F_{p, q}(x), F_{q, r}(y))$

Definition (1.4): Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASx_n, SAx_n}(\varepsilon) \to 1$ for all $\varepsilon > 0$ when ever $\{x_n\}$ is a sequence in x such that $Ax_n, Sx_n \to u$, for some u in X, as $n \to \infty$.

Definition (1.5): Self-maps A and S of a Menger space (X, F, t) are said to be weak compatible (or coincidentally commuting) if they commute at their coincidence points i.e. if Ap = Sp for some $p \in N$ then ASp = SAp.

Lemma [1]: Let $\{p_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norm and $t(x, x) \geq x$. Suppose, for all $x \in [0, 1]$, $\exists K \in (0, 1)$ such that for all $x > 0$ and $n \in N$. $F_{P_n, P_{n+1}}(Kx) \geq F_{P_{n-1}, P_n}(x)$. Then $\{p_n\}$ is a Cauchy sequence in X.

Lemma [2]: Let (X, F, t) be a Menger space, if there exists $K \in (0, 1)$ such that for $p, q \in X$,

$F_{p, q}(Kx) \geq F_{p, q}(x)$, Then p = q.

2. Main Results:

Theorem (2.1): Let A, B, S, T, L, M, P and Q are self mappings on a complete Menger space (X, F, min) satisfying:

(2.1.1) $P(X) \subseteq ST(X) \cup L(X) \cup M(X)$, $Q(X) \subseteq AB(X)$.
(2.1.2) AB = BA, ST = TS, PB = BP, QT = TQ, LT = TL, MT = TM.
(2.1.3) Either P is continuous or AB is continuous.
(2.1.4) (P, AB) is compatible and (L, ST), (Q, ST) and (L, M) are weak compatible.
(2.1.5) There exists $K \in (0, 1)$ such that
$F_{Pu, Qv}(Kx) \geq \min\{F_{ABu, Lv}(x), F_{STv, Pu}(x), F_{STv, Lv}(x), F_{Pu, ABu}(x), F_{ABu, STv}(x), F_{Pu, Mv}(x), F_{Pu, Lv}(x), F_{Qv, STv}(x)\}$ for all $u, v \in X$ and $x > 0$. Then self-maps A, B, S, T, L, M, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in X$, from condition (2.1.1) there exists $x_1, x_2 \in X$ such that

$Px_0 = STx_1 = Lx_1 = Mx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X. such that

$Px_{2n} = STx_{2n+1} = Lx_{2n+1} = Mx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ……

Now we prove $\{y_n\}$ is a Cauchy sequence in X.

Putting $u = x_{2n}$, $v = x_{2n+1}$ for $x > 0$ in (2.1.5) we get,

$$F_{Px_{2n}, Qx_{2n+1}}(Kx) \geq \min\{F_{ABx_{2n}, Lx_{2n+1}}(x), F_{STx_{2n+1}, Px_{2n}}(x), F_{STx_{2n+1}, Lx_{2n+1}}(x), F_{Px_{2n}, ABx_{2n}}(x), F_{ABx_{2n}, STx_{2n+1}}(x), F_{Px_{2n}, Mx_{2n+1}}(x), F_{Px_{2n}, Lx_{2n+1}}(x), F_{Qx_{2n+1}, STx_{2n+1}}(x)\}$$

$$F_{y_{2n}, y_{2n+1}}(Kx) \geq \min\{F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n}}(x), F_{y_{2n}, y_{2n}}(x), F_{y_{2n}, y_{2n-1}}(x), F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n}}(x), F_{y_{2n}, y_{2n}}(x), F_{y_{2n+1}, y_{2n}}(x)\}$$

Hence, $F_{y_{2n}, y_{2n+1}}(Kx) \geq \min\{F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x)\}$

Similarly, $F_{y_{2n+1}, y_{2n+2}}(Kx) \geq \min\{F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n+1}, y_{2n+2}}(x)\}$

Therefore for all n even or odd we have

$$F_{y_n, y_{2n+1}}(Kx) \geq \min\{F_{y_{n-1}, y_n}(x), F_{y_n, y_{n+1}}(x)\}$$

Consequently, $F_{y_n, y_{n+1}}(x) \geq \min\{F_{y_{n-1}, y_n}(k^{-1}x), F_{y_n, y_{n+1}}(k^{-1}x)\}$

By repeated application of above inequality we get,

$$F_{y_n, y_{n+1}}(x) \geq \min\{F_{y_{n-1}, y_n}(k^{-1}x), F_{y_n, y_{n+1}}(k^{-m}x)\}$$

Since $F_{y_n, y_{n+1}}(k^{-m}x) \to 1$ as $n \to \infty$ it follows that

$F_{y_n, y_{n+1}}(kx) \geq F_{y_{n-1}, y_n}(x)$ $\forall$ n $\in$ N and x > 0.

Therefore by Lemma (1), $\{y_n\}$ is a Cauchy sequence in X, which is complete. Hence $\{y_n\} \to z \in X$. Also its subsequences.

$Qx_{2n+1} \to z$, $Lx_{2n+1} \to z$, $STx_{2n+1} \to z$ and $Mx_{2n+1} \to z$ ….. (1)

$Px_{2n} \to z$ and $ABx_{2n} \to z$ ….. (2)

Case 1: When P is continuous: As P is continuous then $PPx_{2n} \to Pz$ and $P(ABx_{2n}) \to Pz$,

As(P, AB) is compatible then we have $P(AB)x_{2n} \to ABz$

As the limit of a sequence in Menger space is unique then we have

ABz = Pz ….. (3)

Step 1: Putting u = z, v = x_{2n+1} in (2.1.5) we get

$$F_{Pz, Qx_{2n+1}}(kx) \geq \text{Min}\{F_{ABz, Lx_{2n+1}}(x), F_{STx_{2n+1}, Pz}(x), F_{STx_{2n+1}, Lx_{2n+1}}(x), F_{Pz, ABz}(x),$$
$$F_{ABz, STx_{2n+1}}(x), F_{Pz, Mx_{2n+1}}(x), F_{Pz, Lx_{2n+1}}(x), F_{Qx_{2n+1}, STx_{2n+1}}(x)\}$$

Letting $n \to \infty$ and using eq^n (1) and (3) we get

$$F_{Pz, z}(Kx) \geq \min\{F_{Pz, z}(x), F_{z, Pz}(x), F_{z, z}(x), F_{Pz, Pz}(x),$$
$$F_{Pz, z}(x), F_{Pz, z}(x), F_{Pz, z}(x), F_{Pz, z}(x), F_{z, z}(x)\}$$

$F_{Pz, z}(Kx) \geq F_{Pz, z}$

By lemma (2) which gives Pz = z. Therefore ABz = Pz = z

Step 2: As $P(X) \subseteq ST(X) \cap L(X) \cap M(X)$ then there exist w $\in$ X such that z = Pz = STw = Lw = Mw.

Putting u = x_{2n}, v = w in (2.1.5) we get,

$$F_{Px_{2n}, Qw}(Kx) \geq \min\{F_{ABx_{2n}, Lw}(x), F_{STw, Px_{2n}}(x), F_{STw, Lw}(x), F_{Px_{2n}, ABx_{2n}}(x),$$
$$F_{ABx_{2n}, STw}(x), F_{Px_{2n}, Mw}(x), F_{Px_{2n}, Lw}(x), F_{Qw, STw}(x)\}$$

Letting $n \to \infty$ using equation (2) we get

$$F_{z, Qw}(Kx) \geq \min\{F_{z, z}(x), F_{z, z}(x), F_{z, z}(x), F_{z, z}(x), F_{z, z}(x),$$
$$F_{z, z}(x), F_{z, z}(x), F_{Qw, z}(x)\}$$

$F_{z, Qw}(Kx) \geq F_{z, Qw}(x)$

Therefore by lemma 2, Qw = z

Hence STw = z = Qw = Lw = Mw

As (Q, ST), (L, ST) and (L, M) are weak compatible, we have

STQw = QSTw, STLw = LSTw and MLw = LMw

Thus STz = Qz, STz = Lz and Mz = Lz

i.e. STz = Qz = Lz = Mz …… (4)

Step 3: Putting u = x_{2n}, v = z in (2.1.5)

$$F_{Px_{2n}, Qz}(Kx) \geq \min\{F_{ABx_{2n}, Lz}(x), F_{STz, Px_{2n}}(x), F_{STz, Lz}(x), F_{Px_{2n} ABx_{2n}}(x),$$
$$F_{ABx_{2n}, STz}(x), F_{Pz_{2n} Mz}(x), F_{Px_{2n}, Lz}(x), F_{Qz, STz}(x)\}$$

letting $n \to \infty$, using eq ^{n} (2) and (4) we get

$$F_{z, Qz}(Kx) \geq \min\{F_{z, Qz}(x), F_{Qz, z}(x), F_{Qz, Qz}(x), F_{z, z}(x), F_{z, Qz}(x), F_{z, Qz}(x),$$
$$F_{z, Qz}(x), F_{Qz, Qz}(x)\}$$

$F_{z, Qz}(Kx) \geq F_{z, Qz}(x)$

Hence z = Qz

Therefore z = Qz = Lz = Mz = STz [from 4]

Step 4: Putting u = x_{2n}, v = Tz in (2.1.5)

$$F_{Px_{2n}, QTz}(Kx) \geq \min\{F_{ABx_{2n}, LTz}(x), F_{ST.Tz, Px_{2n}}(x), F_{ST.Tz, LTz}(x),$$
$$F_{Px_{2n}, ABx_{2n}}(x), F_{ABx_{2n}, ST.Tz}(x), F_{Px_{2n}, MTz}(x),$$
$$F_{Px_{2n} LTz}(x), F_{QTz, ST.Tz}(x)\}$$

As QT = TQ, ST = TS, LT = TL and MT = TM we have

QTz = TQz = Tz, LTz = TLz = Tz, MTz = TMz = Tz

and ST(Tz) = TS(Tz) = T(STz) = Tz

letting $n \to \infty$ and using eq^{n} (2) we get

$$F_{z,Tz}(Kx) \geq \text{Min}\{F_{z,Tz}(x), F_{Tz,z}(x), F_{Tz,Tz}(x), F_{z,z}(x), F_{z,Tz}(x), F_{z,Tz}(x), F_{z,Tz}(x), F_{Tz,Tz}(x)\}$$

$$F_{z,Tz}(Kx) \geq F_{z,Tz}(x)$$

Therefore by lemma (2) we get

$$Tz = z$$

Now $STz = Tz = z$ implies $Sz = z$

Hence $Sz = Tz = Qz = Lz = Pz = Mz = z$ …(α)

Step 5: Putting $u = Bz$, $v = x_{2n+1}$ in (2.1.5), we get

$$F_{PBz,Qx_{2n+1}}(Kx) \geq \min\{F_{AB.Bz,Lx_{2n+1}}(x), F_{STx_{2n+1},PBz}(x), F_{STx_{2n+1},Lx_{2n+1}}(x), F_{PBz,AB.Bz}(x), F_{AB.Bz,STx_{2n+1}}(x), F_{P.Bz,Mx_{2n+1}}(x), F_{PBz,Lx_{2n+1}}(x), F_{Qx_{2n+1},STx_{2x+1}}(x)\}$$

As $PB = BP$ and $AB = BA$. So we have

$PBz = BPz = Bz$ and $AB(Bz) = BA(Bz) = B(ABz) = Bz$

letting $n\to\infty$ and using eq^{n} (1) we get,

$$F_{Bz,z}(Kx) \geq \min\{F_{Bz,z}(x), F_{z,Bz}(x), F_{z,z}(x), F_{Bz,Bz}(x), F_{Bz,z}(x), F_{Bz,z}(x), F_{Bz,z}(x), F_{z,z}(x)\}$$

$$F_{Bz,z}(Kx) \geq F_{Bz,z}(x)$$

which gives $Bz = z$ and $ABz = z$ implies $Az = z$

Therefore $Az = Bz = Pz = z$ .....(β)

Combining (α) and (β) we have

$Az = Bz = Pz = Lz = Qz = Tz = Sz = Mz = z$ i.e. z is the common fixed point of the eight mappings A, B, S, T, L, M, P and Q in the case I.

Case II. AB is continuous:

AS AB is continuous, $AB.ABx_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$.

AS (P, AB) is compatible we have $P(AB)x_{2n} \to ABz$

Step 6: Putting $u = ABx_{2n}$, $v = x_{2n+1}$ in (2.1.5) we get

$$F_{PABx_{2n},Qx_{2n+1}}(kx) \geq \min\{F_{AB.ABx_{2n},Lx_{2n+1}}(x), F_{STx_{2n+1},PABx_{2n}}(x), F_{STx_{2n+1},Lx_{2n+1}}(x), F_{P.ABx_{2n},AB.ABx_{2n}}(x), F_{AB.ABx_{2n},STx_{2n+1}}(x), F_{P.ABx_{2n},Mx_{2n+1}}(x), F_{P.ABx_{2n},Lx_{2n+1}}(x), F_{Qx_{2n+1},STx_{2n+1}}(x)\}$$

Letting $n \to \infty$ we get

$$F_{ABz,z}(kx) \geq \min\{F_{ABz,z}(x), F_{z,ABz}(x), F_{z,z}(x), F_{ABz,ABz}(x), F_{ABz,z}(x), F_{ABz,z}(x), F_{ABz,z}(x), F_{z,z}(x)\}$$

i.e. $F_{ABz,z}(kx) \geq F_{ABz,z}(x)$. Therefore by lemma (2) we get

$$ABz = z$$

Now applying step 1, we get $Pz = z$

Therefore $ABz = Pz = z$

Again applying step 5, to get $Bz = z$ and we get $Az = Pz = Bz = z$.

Now using steps 2, 3 and 4 of previous case we get

$Sz = Tz = Qz = Lz = Mz = z$.

i.e. z is the common fixed point of the eight mappings A, B, S, T, M, L, P and Q in the case II also.

Uniqueness: Let z' be another common fixed point of A, B, S, T, L, M, P and Q then $Az' = Bz' = Sz' = Tz' = Lz' = Mz' = Pz' = Qz' = z'$.

Putting $u = z$, $v = z'$ in (2.1.5) we get

$$F_{Pz,Qz'}(Kx) \geq \min\{F_{ABz,Lz'}(x), F_{STz',Pz}(x), F_{STz',Lz'}(x), F_{Pz,ABz}(x), F_{ABz,STz'}(x), F_{Pz,Mz'}(x), F_{Pz,Lz'}(x), F_{Qz',STz'}(x)\}$$

$$F_{z,z'}(Kx) \geq \min\{F_{z,z'}(x), F_{z',z}(x), F_{z',z}(x), F_{z,z}(x), F_{z,z'}(x), F_{z,z'}(x)\ F_{z,z'}(x), F_{z',z'}(x)\}$$

$$F_{z,z'}(Kx) \geq F_{z,z'}(x)$$

which gives by lemma (2) $z = z'$. Therefore z is a unique common fixed point of A, B, S, T, L, M, P and Q.

In the support of the theorem, we have following example:

Example:- Let (x, F, *) be a Menger space with X = {0,1}, t-norm * defined by a * b = min (a, b), a, b $\in$ [0, 1] and $F_{u,v}(x) = [\exp(|u - v| / x)]^{-1}$, $\forall$ u, v $\in$ X, x > 0. Defined self maps A, B, S, T, L, M, P and Q such that $Pu = \frac{u}{16}$, $Su = \frac{3u}{4}$, $Qu = \frac{u}{32}$, $Tu = \frac{u}{12}$,

$Au = \frac{u}{4}$, $Lu = \frac{u}{3}$, $Bu = \frac{u}{2}$, $Mu = \frac{2u}{3}$, Then for $K \in \left[\frac{1}{2}, 1\right]$

$$F_{Pu,\, Qv}(Kx) = \left[\exp\left(\left|\frac{u}{16} - \frac{v}{32}\right|\right)/Kx\right]^{-1}$$

$$= \left[\exp\left(\left|\frac{u}{8} - \frac{v}{16}\right|\right)/x\right]^{-1}$$

$$= F_{ABu,\, STv}(x)$$

$F_{Pu,\, Qv}(Kx) \geq \min\{F_{ABu,\, Lv}(x), F_{STv,\, Pu}(x), F_{STv,\, Lv}(x), F_{Pu,\, ABu}(x), F_{ABu,\, STv}(x),$

$F_{Pu,\, Mv}(x), F_{Pu,\, Lv}(x), F_{Qv,\, STv}(x)\}$ for all u, v $\in$ X and x > 0.Then all the conditions of theorem are satisfied and zero is the common fixed point of mappings A, B, S, T, L, M, P and Q

References:

[1]. Menger, K.: Statistical metrics, **Proc. Nat. Acad. Sci. USA.,** 28, (1942) 535-537.

[2]. Mishra, S.N.: Common fixed points of compatible mappings in PM-spaces, **Math. Japonica** 36, 2 (1991) 283-289.

[3]. Schweizer, B. and Sklar A.: Statistical metric spaces, **Pacific J. Math.** 10, (1960) 313-334.

[4]. Saini R.K. and Chauhan S.S.: Fixed point theorem for eight mappings on metric spaces, **Journal of Mathematics and System Science,** vol. 1 No. 2 (2005).

[5]. Sehgal,V.M.and Bharucha-Reid,A.T.:Fixed point of contraction mappings in PM-space,**Math.Systems Theory,6**(1972),97-102

[6]. Singh Bijendra and Sharma R.K. Common fixed points of compatible maps in Menger spaces, **Vikram Mathematical Journal**, Vol. XVI, (1996) 51-56.

[7]. Singh, B. and Jain Shishir: A fixed point theorem in Menger space through weak compatibility, **Journal of Mathematical Analysis and Application** 301, (2005) 439-448.

[8]. Singh Bijendra and Jain Shishir: Fixed point theorems in Menger space through semi compatibility,. **Varahmihir Journal** 1 (2006) 117-128

[9]. Singh,S.L.and Singh,S.P: A fixed point theorem,**Indian j.of pure and App. Math**,vol no.11,(1980),1584-1586