*Research Inventy: International Journal of Engineering And Science*
*Vol.13, Issue 9 (September 2023), PP 48-50*
*Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com*

# Study of groups in a metric space

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***Abstract:*** *We proceed to study the algebraic properties of a group, which is inside a metric space. We define a type of algebraic environment from the algebraic operation of the group and the metric defined in the metric space. We analyze some properties of the algebraic environment. The presented algebraic environment can be studied from the metric topology*

***Keywords:*** *Group, metric space, environment.*

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Date of Submission: 02-09-2023 Date of Acceptance: 14-09-2023

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## I. INTRODUCTION

### 1.1 Algebraic Structure: Groups

The study of algebraic structures in mathematics began in the 19th century with the pioneering work of Évariste Galois. From that time on, algebra stopped looking for solutions to equations and turned its attention to the study of algebraic structures. It had great advances in the 19th and 20th centuries.

In modern algebra a group is an algebraic structure that is defined as the dual (G,*) where G is a non-empty set and * is a binary operation that satisfies the following:

1) Closure

Whenever a, b are elements of G, it follows that a*b is also an element of G, so G is closed under internal operation.

$$\forall a, b \in G \qquad a * b \in G$$

2) Associative

For any three elements a, b, c belonging to G the following is true.

$$\forall a, b, c \in G \qquad a * (b * c) = (a * b) * c$$

3) Neutral

There exists an element e in the set G, such that if one operates e with any element a of the set G, the result of the operation is the element a. The element e is called the neutral element.

$$\exists e \in G : \forall a \in G, \qquad a * e = a$$

4) Inverse

For every element a of the set G there exists another element a' such that when a*a' is operated on, the result obtained is the neutral e. The element a' is called the inverse of a.

$$\forall a \in G, \exists a' \in G : a * a' = e$$

Some examples of groups are:

Integers with addition: the sum of 2 integers is another integer, so the operation sum is internal. The sum is associative, the neutral of the sum is 0 and the inverse of p is -p. Under the sum operation, rational numbers, real numbers and complex numbers are also groups.

Rational numbers without 0, with multiplication: the multiplication of 2 rationals is another rational, so it is internal. Multiplication is associative, the neutral is 1 and the inverse of a/b is b/a. Real and complex numbers are also groups with the multiplication operation ([1], [2], [4], [6], [7]).

### 1.2 Metric Space

On the other hand a metric space is defined as the pair (X,d) where X is a non-empty set and d is a real function defined on X×X, called distance or metric, and satisfying the following axioms.

i. $d(x, y) \geq 0 \quad \forall x, y \in X$
ii. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$
iii. $d(x, y) = d(y, x) \quad \forall x, y \in X$
iv. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Some examples of metrics are:

The usual or Euclidean metric between x and y, is defined as the length of the hypotenuse of the right triangle defined by those two points, ie.

$$d_{usual}(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The mail metric between x and y is defined as the sum of the usual distances of both points to the origin of coordinates:

$$d_{cor}(x,y) = d_{usual}(x,0) + d_{usual}(0,y)$$

The taxi or Manhattan metric is defined as the sum of the absolute values of the differences between the coordinates.

$$d_{taxi}(x,y) = |x_1 - x_2| + |y_1 - y_2|$$

The chess metric or maximum between x and y is defined as the maximum of the absolute values of the difference between coordinates ([3], [5], [8]).

$$d_{máx}(x,y) = \{|x_1 - x_2|, |y_1 - y_2|\}$$

## II. Group M

A group M is defined as the third (G,d,*) where G is a nonempty set, d is a metric and * is a binary operation satisfying the following:

a) (G,d) is a metric space.
b) (G,*) is a group.

Given an element a of the set G, we define the algebraic environment of a $E_a$ as the set of elements of G satisfying the following:

$$E_a = \{b \in G | d(a,b) \geq d(a, a * b)\}$$

Properties of the algebraic environment $E_a$

- Property 1

Let e be the neutral of the binary operation *. For any element a of G, e is element of the algebraic environment of a $E_a$

Proof.

Let a be an arbitrary element of G. To verify that e is an element of $E_a$ we proceed to use the definition of the algebraic environment of a. From the definition we have to prove that the distances d(a,e),d(a,a*e) satisfy the following: d(a,e)≥d(a,a*e)

By i) we have

$$d(a,e) \geq 0$$

By ii)

$$d(a,a) = 0$$

From both expressions

$$d(a,e) \geq d(a,a)$$

By the property of the neutral element 3)

$$d(a,e) \geq d(a, a * e)$$

Which is what we wanted to prove, since a is an arbitrary element of G, the result obtained is valid for the other elements of G, therefore e is an element of $E_a$ for every element a of G.

- Property 2

If a is an element of $E_a$, then a^2=a

Proof.

Since a is an element of $E_a$, it satisfies

$$d(a,a) \geq d(a, a * a)$$

Performing the operation * and using ii) we have

$$0 \geq d(a, a^2)$$

By ii) d(a,a^2) is only 0, if a=a^2 which is what we wanted to prove.

- Property 3

If a' is an element of $E_a$, then d(a,a')≥d(a,e)

Proof.

Since a' is an element of $E_a$, it satisfies

$$d(a,a') \geq d(a, a * a')$$

By 4)

$$d(a,a') \geq d(a,e)$$

Which is what was intended to be demonstrated.

- Property 4

If e is the neutral element of *, then $E_e$=G

Proof.

From the definition of $E_e$

$$E_e = \{b \in G | d(e,b) \geq d(e, e * b)\}$$

One has that $E_e$ is a set of elements of G satisfying the metric property, therefore $E_e$ is a subset of G.

To show that G is contained in $E_e$ it suffices to take an arbitrary element a of G and show that it is an element of $E_e$.

Let a be an element of G, to prove that a is an element of $E_e$ it is necessary to show that it satisfies

$$d(e, a) \geq d(e, e * a)$$

Since e is the neutral of * by 3) for every element a of G we have

$$e * a = a$$

Therefore

$$d(e, e * a) = d(e, a)$$

Expression that satisfies

$$d(e, a) \geq d(e, e * a)$$

This is what we wanted to demonstrate.

- Property 5

If a=a^2, then a is element of $E_a$

Proof.

From the definition of $E_a$ it is observed that if we prove d(a,a)=d(a,a*a) is or implies that a is an element of $E_a$.

Since a=a^2 then by ii)

$$0 = d(a, a^2)$$

By ii)

$$d(a, a) = d(a, a^2)$$

Then a^2=a*a therefore

$$d(a, a) = d(a, a * a)$$

Which is what was intended to be demonstrated.

## III. CONCLUSION

In this article the group M was defined as a group which is in turn a metric space. In the same way we proceeded to define the algebraic environment from the binary operation of the group and the metric of the metric space.

We also proceeded to study some of the algebraic properties of the algebraic environment.

From what has been presented, we can continue with a study of the topological properties of the algebraic environment, and the comparison of the balls defined in topology with the algebraic environments defined here.

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